

Existence of Minimizing Willmore Surfaces of Prescribed Conformal Class

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1 Introduction

The Weierstraß formula describes conformal minimal immersions of a Riemann surface into the 3-dimensional Euclidean space in terms of a spinor in the kernel of the Dirac operator on the Riemann surface. Before the Dirac operators was invented, a local generalization of this formula, which is now called Weierstraß representation, was already known by Eisenhardt [Ei]. The global version describes a conformal immersion of a surface into the 3-dimensional Euclidean space again in terms of a spinor in the kernel of the Dirac operator with potential on the surface [Kon, Ta-1, Ta-2, Fr-2]. Pinkall and Pedit generalized this Weierstraß representation to immersion into 4-dimensional Euclidean space and invented the ‘quaternionic function theory’ [P-P, B-F-L-P-P, F-L-P-P]. From their point of view conformal immersion of Riemann surfaces into the 4-dimensional Euclidean space (identified with the quaternions) are essentially sections of holomorphic quaternionic line bundles. These holomorphic quaternionic line bundles are build form an usual holomorphic complex line bundle on the Riemann surface together with a Hopf field. Due to an observation of Taimanov, the Willmore functional is equal to four times the integral over the square of the potential [Ta-1]. Our main subject is the investigation of these holomorphic quaternionic line bundles, whose Hopf fields are square integrable. The holomorphic sections of such quaternionic line bundles form the maximal domain of definition of the Willmore functional on the space of conformal mappings of a Riemann surface into $\mathbb{H} \simeq \mathbb{R}^4$. In the second section we extend Cauchy’s integral formula to these holomorphic section of quaternionic holomorphic line bundles. In the fourth section we show that the corresponding sections define sheaves, and that the Čech cohomology groups of these sheaves obey the Riemann–Roch Theorem and Sérre duality. In the fifth section extend those Bäcklund transformations to square integrable Hopf fields, which relate the infinitesimal quaternionic Weierstraß representation to the Kodaira embedding of ‘quaternionic function theory’ [P-P]. This yields in the sixth section a general proof of the Plücker formula [F-L-P-P] for these holomorphic quaternionic line bundles with square integrable Hopf fields. In the seventh section we show that any bounded sequence of square integrable Hopf fields has a convergent subsequence, and that the limit is again the Hopf field of a holomorphic quaternionic line bundle, but the holomorphic structure might have singularities. This is used in the last section to proof that the Willmore functional has on the space of all conformal immersions of a compact Riemann surface into the 3-dimensional and 4-dimensional Euclidean space a minimum. Moreover, even the restrictions of the Willmore functional to all conformal immersions into the 4-dimensional Euclidean space has a minimum, whose underlying holomorphic complex line bundle (compare [P-P, F-L-P-P]) is fixed. The existence of a minimizer on the space of all immersions from a Riemann surface of prescribed genus into the n —dimensional Euclidean spaces was proven by Simon for genus one [Si-1, Si-2], and recently by Bauer and Kuwert for all finite genera [B-K].

We identify the quaternions with all complex 2×2 -matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$. If we consider a \mathbb{C}^2 -valued function $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ on an open set $\Omega \subset \mathbb{C}$ as a quaternionic valued function $\begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$, then the operator $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ defines a quaternionic holomorphic structure in the sense of [F-L-P-P, Definition 2.1.] on the trivial quaternionic line bundle on Ω endowed

with the complex structure of multiplication on the left with complex numbers $\mathbb{C} \subset \mathbb{Q}$. In particular, the action of $\sqrt{-1}$ is given by left-multiplication with $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. The corresponding holomorphic sections are defined as the elements of the kernel of this operator, which agrees with the elements of the kernel of the Dirac operator

$$\begin{pmatrix} U & \partial \\ -\bar{\partial} & \bar{U} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}.$$

The corresponding Hopf field is equal to $Q = -\bar{U}d\bar{z}$. The space of holomorphic sections is invariant under right-multiplication with quaternions and therefore a quaternionic vector space.

The holomorphic structure is an operator from the sections of a quaternionic line bundle into the space of sections of this quaternionic line bundle tensored with the line bundle of anti-holomorphic forms ($\simeq \mathcal{O}_{-K}$) [F-L-P-P, §2.2]. We represent the underlying holomorphic complex line bundle on a Riemann surface X as the trivial complex line bundles on all members of an open covering together with a cocycle in the corresponding multiplicative first Čech Cocomplex, which represents an element in $H^1(X, \mathcal{O}^*)$. We shall state how the holomorphic structure transforms under these cocycles and coordinate transformations $z \mapsto z' = z'(z)$. The multiplication with a non-vanishing function f acts on the spinors as $\psi \mapsto \begin{pmatrix} f & 0 \\ 0 & \bar{f} \end{pmatrix} \psi$. Therefore this multiplicative cocycle acts on the holomorphic structure as

$$\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \mapsto \begin{pmatrix} f & 0 \\ 0 & \bar{f} \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & \bar{f} \end{pmatrix}^{-1} = \begin{pmatrix} -\bar{\partial} & -\frac{f}{\bar{f}}\bar{U} \\ \frac{f}{\bar{f}}U & \partial \end{pmatrix}.$$

The corresponding potential U and Hopf field Q transforms as $U \mapsto \frac{f}{\bar{f}}U$ and $Q \mapsto \frac{f}{\bar{f}}Q$. The coordinate transformation $z \mapsto z' = z'(z)$ acts on the holomorphic structure as

$$\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \mapsto \begin{pmatrix} \bar{\partial}' & -\bar{U}' \\ U' & \partial' \end{pmatrix} = \begin{pmatrix} \frac{dz'}{dz} & 0 \\ 0 & \frac{d\bar{z}'}{d\bar{z}} \end{pmatrix}^{-1} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} = \begin{pmatrix} \bar{\partial}' & -\frac{dz'}{d\bar{z}'}\bar{U} \\ \frac{dz'}{dz}U & \partial' \end{pmatrix}.$$

Therefore the potentials transforms as $U \mapsto U' = \frac{dz}{dz'}U$ and the corresponding Hopf field $Q = -\bar{U}d\bar{z} = -\bar{U}'d\bar{z}'$ does not change. Summing up, for any holomorphic complex line bundle, which is represented by the trivial line bundles on all members of an open covering together with a cocycle in $H^1(X, \mathcal{O}^*)$, this cocycle defines also cocycles for the corresponding spinors ψ and potentials U .

Quaternionic Weierstraß Representation 1.1. *[P-P, Theorem 4.3.] For any conformal immersion $f : X \rightarrow \mathbb{H}$ of a Riemann surface X there exist two quaternionic holomorphic line bundles with two holomorphic sections ψ and ϕ , such that the derivative of f is given by*

$$d \begin{pmatrix} f_1 & -\bar{f}_2 \\ f_2 & \bar{f}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix} \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} \text{ with}$$

$$\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = 0 \quad \begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix} \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix} = 0.$$

q.e.d.

We remark that the product of the underlying complex line bundles has to be equal to the anti-canonical line bundle (i. e. the line bundle of anti-holomorphic forms) and that the potentials of both holomorphic structures are determined by each other. Immersion into \mathbb{R}^3 are obtained as immersion into the pure imaginary quaternions $\simeq \mathbb{R}^3$. This is realized by the additional reality conditions $\bar{U} = U$, $df^* = -df$

$$\begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix}^* = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}^* = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

2 Local behaviour of holomorphic spinors

Dolbeault's Lemma [Gu-Ro, Chapter I Section D 2. Lemma] implies that the operator $\mathbf{l}_{\mathbb{C}}(0)$ with the integral kernel

$$\begin{pmatrix} (z - z')^{-1} & 0 \\ 0 & (\bar{z} - \bar{z}')^{-1} \end{pmatrix} \frac{d\bar{z}' \wedge dz'}{2\pi\sqrt{-1}}$$

is a right inverse of the operator $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}$. Due to the Hardy–Littlewood–Sobolev theorem [St, Chapter V. §1.2 Theorem 1] for all $1 < p < 2$ and $2 < q < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$ this is a bounded operator from $L^p(\mathbb{C}, \mathbb{H})$ into $L^q(\mathbb{C}, \mathbb{H})$. Moreover, the restriction $\mathbf{l}_{\Omega}(0)$ of $\mathbf{l}_{\mathbb{C}}(0)$ to a bounded open domain Ω is a bounded operator from $L^p(\Omega, \mathbb{H})$ into $L^q(\Omega, \mathbb{H})$. On the other hand Hölder's inequality [R-S-I, Theorem III.1 (c)] implies that the multiplication operators with $U \in L^2(\Omega)$ are bounded operators from $L^q(\Omega)$ into $L^p(\Omega)$. Hence the operator $\mathbf{1} + \mathbf{l}_{\Omega}(0) \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix}$ is a bounded operator on $L^q(\Omega, \mathbb{H})$. For smooth U a spinor ψ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$, if and only if $\mathbf{1} + \mathbf{l}_{\Omega}(0) \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix}$ maps ψ into the kernel of $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}$. Due to Weyl's Lemma [R-S-II, Theorem IX.25] all elements in the kernel of this differential operator are smooth functions. Consequently, for all $U \in L^2_{\text{loc}}(\Omega)$ the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ is defined as all spinors $\psi \in L^q_{\text{loc}}(\Omega, \mathbb{H})$, which are mapped by $\mathbf{1} + \mathbf{l}_{\Omega}(0) \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix}$ into the kernel of $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}$. Finally, we remark that we may always cover Ω by small sets Ω' such that the von Neumann series

$$\left(\mathbf{1} + \mathbf{l}_{\Omega}(0) \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix} \right)^{-1} = \sum_{l=0}^{\infty} \left(\mathbf{l}_{\Omega}(0) \begin{pmatrix} 0 & \bar{U} \\ -U & 0 \end{pmatrix} \right)^l$$

converges as an operator on $L^q(\Omega', \mathbb{H})$, which maps the closed subspace of bounded elements in the kernel of $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}$ onto the closed subspace of bounded elements in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$. Therefore the latter kernel is contained in $\bigcap_{1 < p < 2} W^{1,p}_{\text{loc}}(\Omega, \mathbb{H}) \subset \bigcap_{q < \infty} L^q_{\text{loc}}(\Omega, \mathbb{H})$.

Moreover, on small domains $\Omega \subset \mathbb{C}$ the operator

$$\mathbf{l}_{\Omega}(U) = \mathbf{l}_{\Omega}(0) \left(\mathbf{1} + \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix} \mathbf{l}_{\Omega}(0) \right)^{-1} = \left(\mathbf{1} + \mathbf{l}_{\Omega}(0) \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix} \right)^{-1} \mathbf{l}_{\Omega}(0)$$

is a right inverse of the operator $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$. If $\mathcal{K}_\Omega(U, z, z') \frac{d\bar{z} \wedge dz}{2\pi\sqrt{-1}}$ denotes the integral kernel of this operator $\mathbf{l}_\Omega(U)$, then we have

$$\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \mathcal{K}_\Omega(U, z, z') = \pi\delta(z - z')\mathbf{1} \quad \begin{pmatrix} -\bar{\partial} & U \\ -\bar{U} & -\partial \end{pmatrix} \mathcal{K}_\Omega^t(U, z', z) = \pi\delta(z - z')\mathbf{1}.$$

Here the differential operator and his transposed acts on the integral kernel as a function depending on z for fixed z' . If ψ is an element of the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and ϕ an element of the kernel of $\begin{pmatrix} -\bar{\partial} & U \\ -\bar{U} & -\partial \end{pmatrix}$, then a direct calculation shows

$$\begin{aligned} d \left(\mathcal{K}_\Omega(U, z', z) \begin{pmatrix} dz & 0 \\ 0 & -d\bar{z} \end{pmatrix} \psi(z) \right) &= \pi\delta(z - z')\psi(z)d\bar{z} \wedge dz \\ d \left(\phi(z) \begin{pmatrix} dz & 0 \\ 0 & -d\bar{z} \end{pmatrix} \mathcal{K}_\Omega(U, z, z') \right) &= \pi\delta(z - z')\phi(z)dz \wedge d\bar{z}. \end{aligned}$$

This implies a quaternionic version of

Cauchy's Integral Formula 2.1. *All elements ψ and ϕ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and $\begin{pmatrix} -\bar{\partial} & U \\ -\bar{U} & -\partial \end{pmatrix}$ on a small open set Ω obey the formula*

$$\begin{aligned} \psi(z') &= \frac{1}{2\pi\sqrt{-1}} \oint \mathcal{K}_\Omega(U, z', z) \begin{pmatrix} dz & 0 \\ 0 & -d\bar{z} \end{pmatrix} \psi(z) \\ \phi(z') &= \frac{-1}{2\pi\sqrt{-1}} \oint \phi(z) \begin{pmatrix} dz & 0 \\ 0 & -d\bar{z} \end{pmatrix} \mathcal{K}_\Omega(U, z, z'), \end{aligned}$$

as long as the integration path surrounds z' one times in the anti-clockwise-order, respectively.

Remark 2.2. *At a first look it is not clear, whether the integral along the closed path is well defined. However, since on the complement of $\{z'\}$ the corresponding one-forms are closed, we may extend the integration over the closed path to an integration over a cylinder around z' . More precisely, let f be a quaternionic smooth function with compact support in Ω , which is equal to $\mathbf{1}$ on an open subset Ω' with $\bar{\Omega}' \subset \Omega$. Then we have the following equality of measurable functions on $z' \in \Omega'$:*

$$\begin{aligned} \psi(z') &= \frac{1}{2\pi\sqrt{-1}} \int_{\Omega} df \wedge \mathcal{K}_\Omega(U, z', z) \begin{pmatrix} dz & 0 \\ 0 & -d\bar{z} \end{pmatrix} \psi(z) \\ \phi(z') &= \frac{-1}{2\pi\sqrt{-1}} \int_{\Omega} \phi(z) \begin{pmatrix} dz & 0 \\ 0 & -d\bar{z} \end{pmatrix} \mathcal{K}_\Omega(U, z, z') \wedge df. \end{aligned}$$

In particular, for all square integrable Hopf fields (i. e. all potentials belong to L^2_{loc}) the holomorphic sections of the corresponding holomorphic quaternionic line bundle define a sheaf on X . In the sequel we shall denote by \mathcal{Q}_D the sheaf of sections of a holomorphic quaternionic line bundle over the complex line bundle corresponding to \mathcal{O}_D .

Lemma 2.3. *For small potentials $U \in L^2(\Omega)$ on a bounded domain Ω there exists positive functions $\mathcal{A}, \mathcal{B} \in \bigcap_{q < \infty} L^q(\Omega)$ such that the integral kernels $\mathcal{K}_\Omega(U, z, z')$ may be estimated by*

$$|\mathcal{K}_\Omega(U, z, z')| \leq \frac{\mathcal{A}(z)\mathcal{B}(z')}{|z - z'|}.$$

Proof. The equation $\frac{1}{(z - z')(z' - z'')} + \frac{1}{(z' - z'')(z'' - z)} + \frac{1}{(z'' - z)(z - z')} = 0$ implies the estimate

$$\frac{1}{|z - z'| |z' - z''|} \leq \frac{1}{|z' - z''| |z'' - z|} + \frac{1}{|z'' - z| |z - z'|}.$$

Therefore the integral kernel of the operator $\mathbf{l}_\Omega(0) \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix} \mathbf{l}_\Omega(0)$ is bounded by $\frac{\mathcal{F}(z) + \mathcal{F}(z')}{\pi^2 |z - z'|}$, where $\mathcal{F}(z)$ is the convolution of the L^2 -function $|U|$ with the positive function $\frac{1}{|z|}$. Due to Young's inequality [R-S-II, Section IX.4 Example 1] this function \mathcal{F} belongs to $\bigcap_{q < \infty} L^q(\Omega)$. An iterative application of this argument to all terms of the von Neumann series of $\mathbf{l}_\Omega(U)$ yields a bound of the integral kernel of the form $\sum_l \frac{\mathcal{A}_l(z)\mathcal{B}_l(z')}{|z - z'|}$. For small L^2 -norms of U all $L^q(\Omega)$ -norms of $\sum_l \mathcal{A}_l$ and $\sum_l \mathcal{B}_l$ are bounded. This completes the proof. **q.e.d.**

The holomorphic sections of a holomorphic quaternionic line bundle are in general not continuous. Nevertheless they share many properties with the sheaves of holomorphic functions. In particular, they have the *Strong unique continuation property*, which is proven by a *Carleman inequality* (compare with [Ca] and [Wo, Proposition 1.3]).

Carleman inequality 2.4. *There exists some constant S_p depending only on $1 < p < 2$, such that for all $n \in \mathbb{Z}$ and all $\psi \in C_0^\infty(\mathbb{C} \setminus \{0\}, \mathbb{H})$ the following inequality holds:*

$$\| |z|^{-n} \psi \|_{\frac{2p}{2-p}} \leq S_p \| |z|^{-n} \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi \|_p.$$

The literature [Je, Ki-1, Ma, Ki-2] deals with the much more difficult higher-dimensional case and does not treat our case. David Jerison pointed out to the author, that the arguments of [Wo, Proposition 2.6], where the analogous but weaker statement about the gradient term of the Laplace operator is treated, carry over to the Dirac operator.

Proof. Dolbeault's Lemma [Gu-Ro, Chapter I Section D 2. Lemma] implies for all smooth ψ with compact support the equality

$$\psi(z) = \int_{\mathbb{C}} \begin{pmatrix} (z - z')^{-1} & 0 \\ 0 & (\bar{z} - \bar{z}')^{-1} \end{pmatrix} \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi(z') \frac{d\bar{z}' \wedge dz'}{2\pi\sqrt{-1}}.$$

In fact, the components of the difference of the left hand side minus the right hand side are holomorphic and anti-holomorphic functions on \mathbb{C} , respectively, which vanish at $z = \infty$. In

particular, the integrals $\int_{\mathbb{C}} z^n \bar{\partial} \psi_1 d\bar{z} \wedge dz$ and $\int_{\mathbb{C}} \bar{z}^n \partial \psi_2 d\bar{z} \wedge dz$ with $n \in \mathbb{N}_0$ are proportional to the Taylor coefficients of ψ at ∞ , which vanish. Moreover, if the support of ψ does not contain 0 and therefore also a small neighbourhood of 0, then the integrals $\int_{\mathbb{C}} z^{-n} \bar{\partial} \psi_1 d\bar{z} \wedge dz$ and $\int_{\mathbb{C}} \bar{z}^{-n} \partial \psi_2 d\bar{z} \wedge dz$ with $n \in \mathbb{N}$ are proportional to the Taylor coefficients of ψ at 0, which in this case also vanish. These cancellations follow also from partial integration. We conclude that for all $n \in \mathbb{Z}$

$$\psi(z) = \int_{\mathbb{C}} \begin{pmatrix} \left(\frac{z}{z'}\right)^n \frac{1}{z-z'} & 0 \\ 0 & \left(\frac{\bar{z}}{\bar{z}'}\right)^n \frac{1}{\bar{z}-\bar{z}'} \end{pmatrix} \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi(z') \frac{d\bar{z}' \wedge dz'}{2\pi\sqrt{-1}}.$$

In fact, for negative n the left hand side minus the left hand side of the foregoing formula is equal to the Taylor polynomial of ψ at ∞ up to order $|n|$, and for positive n equal to the Taylor polynomial of ψ at 0 up to order $n-1$. Finally, the Hardy–Littlewood–Sobolev theorem [St, Chapter V. §1.2 Theorem 1] implies that the operator with integral kernel

$$\begin{pmatrix} \left(\frac{|z'|z}{|z|\bar{z}'}\right)^n \frac{1}{z-z'} & 0 \\ 0 & \left(\frac{|z'|\bar{z}}{|z|\bar{z}'}\right)^n \frac{1}{\bar{z}-\bar{z}'} \end{pmatrix} \frac{d\bar{z}' \wedge dz'}{2\pi\sqrt{-1}}$$

from $L^p(\mathbb{C}, \mathbb{H})$ into $L^{\frac{2p}{2-p}}(\mathbb{C}, \mathbb{H})$ is bounded by some constant S_p not depending on n , and maps $|z|^{-n} \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi$ onto $|z|^{-n} \psi$. **q.e.d.**

Due to a standard argument (e. g. [So, Proof of Theorem 5.1.4] and [Wo, Section Carleman Method]) this *Carleman inequality* implies the

Strong unique continuation property 2.5. *Let U be a potential in $L^2_{\text{loc}}(\Omega)$ on an open connected set $0 \ni \Omega \subset \mathbb{C}$ and $\psi \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{H})$ an element of the kernel of $\begin{pmatrix} \bar{\partial} & -U \\ U & \partial \end{pmatrix}$ on Ω with $1 < p < 2$. If the $L^{\frac{2p}{2-p}}$ -norm of the restriction of ψ to the balls $B(0, \varepsilon)$ converges in the limit $\varepsilon \downarrow 0$ faster to zero than any power of ε :*

$$\left(\int_{B(0, \varepsilon)} |\psi|^{\frac{2p}{2-p}} d^2x \right)^{\frac{2-p}{2p}} \leq \mathbf{O}(\varepsilon^n) \quad \forall n \in \mathbb{N},$$

then ψ vanishes identically on Ω .

Proof. The question is local so we may assume that U is an element of L^2 rather than L^2_{loc} . We fix ε small enough that $\|U\|_{L^2(B(z, 2\varepsilon))} \leq 1/(2S_p)$ for all z , where S_p is the constant of the *Carleman inequality*. Let $\phi \in C^\infty$ be 1 on $B(0, \varepsilon)$ and 0 on $\mathbb{C} \setminus B(0, 2\varepsilon)$. A limiting argument using the infinite order vanishing of ψ and the equality $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix} \psi$ shows that the proof of the *Carleman inequality* is also true for $\phi\psi$. So

$$\||z|^{-n} \phi\psi\|_{\frac{2p}{2-p}} \leq S_p \||z|^{-n} \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \phi\psi\|_p \leq S_p \||z|^{-n} \phi \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi\|_p + S_p \||z|^{-n} E\|_p.$$

Here E (for error) $= \begin{pmatrix} \psi_1 \bar{\partial} \phi_1 \\ -\psi_2 \bar{\partial} \phi_2 \end{pmatrix}$ is an L^p function supported in $\{x \mid \varepsilon \leq |z| \leq 2\varepsilon\}$. Using the equality $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \bar{\partial} \end{pmatrix} \psi = \begin{pmatrix} 0 & \bar{U} \\ -U & 0 \end{pmatrix} \psi$ and Hölder's inequality [R-S-I, Theorem III.1 (c)] yields

$$\begin{aligned} \| |z|^{-n} \phi \psi \|_{\frac{2p}{2-p}} &\leq S_p \| |z|^{-n} \phi \begin{pmatrix} 0 & \bar{U} \\ -U & 0 \end{pmatrix} \psi \|_p + S_p \| |z|^{-n} E \|_p \\ &\leq S_p \| U \|_{L^2(B(0, 2\varepsilon))} \| |z|^{-n} \phi \psi \|_{\frac{2p}{2-p}} + S_p \| |z|^{-n} E \|_p. \end{aligned}$$

By the choice of ε the first term can be absorbed into a factor 2

$$\| |z|^{-n} \phi \psi \|_{\frac{2p}{2-p}} \leq 2S_p \| |z|^{-n} E \|_p.$$

Now comes the crucial observation: E is supported in $\{z \mid \varepsilon \leq |z| \leq 2\varepsilon\}$, so

$$\| |z|^{-n} \phi \psi \|_{\frac{2p}{2-p}} \leq 2S_p \varepsilon^{-n} \| E \|_p \quad \text{and} \quad \left\| \left(\frac{\varepsilon}{|z|} \right)^n \phi \psi \right\|_{\frac{2p}{2-p}} \leq 2S_p \| E \|_p.$$

Using the limit $n \rightarrow \infty$ we conclude that $\phi \psi$ vanishes on $B(0, \varepsilon)$, and therefore also ψ . In other words, the set $\{z \mid \psi \text{ vanishes to infinite order at } z\}$ is open, and in fact contains a ball of fixed radius ε centered at any of its points. So this set must be all of Ω and the proof is complete. **q.e.d.**

The definition of the order of a zero extends from the complex case to the quaternionic case.

Order of zeroes 2.6. *The order of a zero of ψ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$ on an open neighbourhood $0 \in \Omega \subset \mathbb{C}$ at $z = 0$ is defined as the largest integer m , such that $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \bar{\partial} \end{pmatrix}^{-m} \psi \in L_{\text{loc}}^q(\Omega, \mathbb{H})$ with $2 < q < \infty$ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U}(\frac{z}{\varepsilon})^m \\ U(\frac{z}{\varepsilon})^m & \bar{\partial} \end{pmatrix}$. Due to the Strong unique continuation property 2.5 this number is finite and denoted by $\text{ord}_0 \psi$.*

On small open domains Ω with small L^2 -norms of the potential U the elements of the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$ are small perturbations of the elements of the kernel of $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \bar{\partial} \end{pmatrix}$. We conclude that the quotient of the space of holomorphic spinors divided by the subspace of holomorphic spinors vanishing at z_0 is a one-dimensional quaternionic vector space. Hence all holomorphic spinors are non-vanishing sections of another holomorphic quaternionic line bundle. Moreover, the zeroes of such ψ are isolated. Furthermore, the divisor of an holomorphic section of a holomorphic quaternionic line bundle is well defined. Furthermore, for any divisor D and any sheaf \mathcal{Q}_D of holomorphic sections of a holomorphic quaternionic line bundle those holomorphic sections, whose divisors are larger than $-D$, define the sheaf of holomorphic sections of another holomorphic quaternionic line bundle, which is denoted by \mathcal{Q}_D . In particular, any holomorphic section of a holomorphic quaternionic line bundle is the non-vanishing holomorphic section of another holomorphic quaternionic line bundle.

For an effective divisor D (i. e. $D \geq 0$) the quotient sheaf $\mathcal{Q}_D/\mathcal{Q}$ has the same support as the divisor D . For the divisor of the function $z \mapsto z^l$ on $0 \ni \Omega \subset \mathbb{C}$ this quotient is

isomorphic to the codimension of the image of the operator

$$\left(\mathbf{1} + \begin{pmatrix} 0 & -(\frac{\bar{z}}{z})^l \bar{U} \\ (\frac{z}{\bar{z}})^l U & 0 \end{pmatrix} \mathbf{l}_{\Omega(0)} \right) \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}^l \left(\mathbf{1} + \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix} \mathbf{l}_{\Omega(0)} \right)^{-1}$$

considered as an operator on the kernel of the free Dirac operator. For small Ω with small $L^2(\Omega)$ -norms of U , this operator is a small perturbation of $(\frac{z}{0} \frac{0}{\bar{z}})^l$. This proves

Lemma 2.7. *For any pair of divisors $D' \geq D$ on a Riemann surface X and any sheaf \mathcal{Q}_D of holomorphic sections of a holomorphic quaternionic line bundle the quaternionic dimension of $H^0(X, \mathcal{Q}_{D'}/\mathcal{Q}_D)$ is equal to $\deg(D' - D)$ and $H^1(X, \mathcal{Q}_{D'}/\mathcal{Q}_D)$ is trivial. **q.e.d.***

3 Spectral theory of Dirac operators

The holomorphic structures of quaternionic line bundles may be described by first order differential operators similar to Dirac operators. In this section we develop in six steps the spectral theory of Dirac operators on compact Riemann surfaces. Due to the Sobolev Embedding [Ad, 5.4 Theorem], these Dirac operators are for all $1 < p < 2$ bounded operators from the Sobolev spaces of $W^{1,p}$ -spinors into the L^p -spinors. Furthermore, their resolvents turn out to be bounded operators from the L^p -spinors onto the $W^{1,p}$ -spinors. Hence we shall define the domains of these Dirac operators, considered as unbounded closed operators on the Hilbert space of L^2 -spinors, as the images in the $W^{1,p}$ -spinors of the L^2 -spinors under the resolvents. The corresponding closed unbounded operators are defined as the restrictions of the Dirac operators from the $W^{1,p}$ -spinors into the L^p -spinors.

1. Uniformization of compact Riemann surfaces. We choose a holomorphic complex line bundle, which is a square root of the canonical bundle. The corresponding holomorphic structures are given by Dirac operators with potentials. In order to develop the spectral theory of these Dirac operators we represent the compact Riemann surfaces of genus larger than one as quotients \mathbb{D}/Γ of the hyperbolic disk \mathbb{D} modulo a Fuchsian group and elliptic curves as quotients \mathbb{C}/Λ of \mathbb{C} modulo a lattice Λ [F-K, Chapter IV.5.]. The corresponding group action has a fundamental domain denoted by Δ [F-K, Chapter IV.9.]. The corresponding spin bundle is an induced bundle of an representation of Γ and Λ , respectively. Therefore we shall consider the Dirac operators and their resolvents on the Riemann sphere \mathbb{P} with the elliptic metric $\frac{dzd\bar{z}}{(1+z\bar{z})^2}$, on the complex plane \mathbb{C} with flat metric $dzd\bar{z}$ and on the hyperbolic disk \mathbb{D} with hyperbolic metric $\frac{dzd\bar{z}}{(1-z\bar{z})^2}$. The corresponding Dirac operators are of the form [Fr-1, Chapter 3.4.]

$$\begin{pmatrix} 0 & (1+z\bar{z})\partial \\ -(1+z\bar{z})\bar{\partial} & 0 \end{pmatrix} \text{ on } \mathbb{P}, \quad \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} \text{ on } \mathbb{C} \text{ and } \begin{pmatrix} 0 & (1-z\bar{z})\partial \\ -(1-z\bar{z})\bar{\partial} & 0 \end{pmatrix} \text{ on } \mathbb{D}.$$

2. Green's functions. We shall calculate the integral kernels of the corresponding resolvents, which are the inverse of the operators

$$\begin{pmatrix} \sqrt{-1}\lambda & -(1+z\bar{z})\partial \\ (1+z\bar{z})\bar{\partial} & \sqrt{-1}\lambda \end{pmatrix}, \quad \begin{pmatrix} \sqrt{-1}\lambda & -\partial \\ \bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{-1}\lambda & -(1-z\bar{z})\partial \\ (1-z\bar{z})\bar{\partial} & \sqrt{-1}\lambda \end{pmatrix}.$$

The composition of these operators with the operators

$$\begin{pmatrix} \sqrt{-1}\lambda & (1+z\bar{z})\partial \\ -(1+z\bar{z})\bar{\partial} & \sqrt{-1}\lambda \end{pmatrix}, \quad \begin{pmatrix} \sqrt{-1}\lambda & \partial \\ -\bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{-1}\lambda & (1-z\bar{z})\partial \\ -(1-z\bar{z})\bar{\partial} & \sqrt{-1}\lambda \end{pmatrix}$$

are equal to the operators

$$\begin{pmatrix} -\lambda^2 + (1+z\bar{z})^2\partial\bar{\partial} + (1+z\bar{z})\bar{z}\bar{\partial} & 0 \\ 0 & -\lambda^2 + (1+z\bar{z})^2\bar{\partial}\partial + (1+z\bar{z})\partial \end{pmatrix},$$

$$\begin{pmatrix} -\lambda^2 + \partial\bar{\partial} & 0 \\ 0 & -\lambda^2 + \bar{\partial}\partial \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} -\lambda^2 + (1-z\bar{z})^2\partial\bar{\partial} - (1-z\bar{z})\bar{z}\bar{\partial} & 0 \\ 0 & -\lambda^2 + (1-z\bar{z})^2\bar{\partial}\partial - (1-z\bar{z})z\partial \end{pmatrix}.$$

On functions, which depend only on $r = |z|$ these operators act as diagonal matrices, whose entries are the operators

$$-\lambda^2 + \left(\frac{1+r^2}{2}\right)^2 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) + \frac{1+r^2}{2}r\frac{d}{dr},$$

$$-\lambda^2 + \frac{1}{4} \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) \text{ and}$$

$$-\lambda^2 + \left(\frac{1-r^2}{2}\right)^2 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) - \frac{1-r^2}{2}r\frac{d}{dr}, \text{ respectively.}$$

the substitutions $r = \tan(y/2)$, $r = y/2$ and $r = \tanh(y/2)$ transforms these operators into

$$-\lambda^2 + \frac{d^2}{dy^2} + \frac{1}{\sin(y)}\frac{d}{dy}, \quad -\lambda^2 + \frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} \text{ and } -\lambda^2 + \frac{d^2}{dy^2} + \frac{1}{\sinh(y)}\frac{d}{dy}.$$

We remark that in all three cases y is twice the distance from the origin. They define self-adjoint operators on the Hilbert spaces corresponding to the measure spaces

$$\frac{\pi \sin(y)dy}{\cos(y)+1} \text{ on } y \in [0, \pi], \quad \frac{\pi y dy}{2} \text{ on } y \in [0, \infty), \quad \frac{\pi \sinh(y)dy}{\cosh(y)+1} \text{ on } y \in [0, \infty).$$

Let $\mathcal{G}_{\mathbb{P},\lambda}$, $\mathcal{G}_{\mathbb{C},\lambda}$ and $\mathcal{G}_{\mathbb{D},\lambda}$ denote the corresponding Green's functions, i. e. the applications of the three operators above on these functions yields the δ -function with respect to the corresponding measures (which are equal to the usual two-dimensional δ -function on the complex plane). Due to [St, Chapter V. §3.1 and §6.5] and [G-J, Section 7.2] the second

function $\mathcal{G}_{\mathbb{C},\lambda}$ is for $y > 0$ given by

$$\begin{aligned}
\mathcal{G}_{\mathbb{C},\lambda}(y) &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp(-\pi\sqrt{-1}yk_1)}{\lambda^2 + \pi^2(k_1^2 + k_2^2)} dk_1 dk_2 \\
&= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp(-\sqrt{-1}|\lambda|yk_1)}{\pi^2(1 + k_1^2 + k_2^2)} dk_1 dk_2 \\
&= - \int_{-\infty}^{\infty} \frac{\exp(-|\lambda|y\sqrt{1+k^2})}{\pi\sqrt{1+k^2}} dk \\
&= - \frac{2}{\pi} \int_1^{\infty} \frac{\exp(-|\lambda|yx)}{\sqrt{x^2-1}} dx \\
&= - \frac{2}{\pi} \int_{|\lambda|y}^{\infty} \frac{\exp(-x)}{\sqrt{x^2-\lambda^2y^2}} dx.
\end{aligned}$$

This implies that this function $\mathcal{G}_{\mathbb{C},\lambda}$ has the following properties

- (i) $0 < -\mathcal{G}_{\mathbb{C},\lambda}(y) \leq \mathbf{O}(1) \exp((\varepsilon - |\lambda|)y)$ with an appropriate $\varepsilon > 0$ and large $|\lambda|y$.
- (ii) $0 < \mathcal{G}'_{\mathbb{C},\lambda}(y) \leq \mathbf{O}(1) \exp((\varepsilon - |\lambda|)y)$ with an appropriate $\varepsilon > 0$ and large $|\lambda|y$.
- (iii) $0 < -\mathcal{G}_{\mathbb{C},\lambda}(y) \leq -\frac{2}{\pi} \ln(|\lambda|y) + \mathbf{O}(1)$ for small y .
- (iv) $0 < \mathcal{G}'_{\mathbb{C},\lambda}(y) \leq \frac{2}{\pi} \frac{1}{y} + \mathbf{O}(1)$ for small y .

The first and the third operator may be transformed into the operators

$$\begin{aligned}
\cos^{-1}\left(\frac{y}{2}\right) \left(\lambda^2 - \frac{d^2}{dy^2} - \frac{1}{\sin(y)} \frac{d}{dy} \right) \cos\left(\frac{y}{2}\right) &= \lambda^2 - \frac{d^2}{dy^2} - \frac{\cos(y)}{\sin(y)} \frac{d}{dy} + \frac{\sin^2\left(\frac{y}{2}\right)}{4\cos^2\left(\frac{y}{2}\right)} \\
\cosh^{-1}\left(\frac{y}{2}\right) \left(\lambda^2 - \frac{d^2}{dy^2} - \frac{1}{\sinh(y)} \frac{d}{dy} \right) \cosh\left(\frac{y}{2}\right) &= \lambda^2 - \frac{d^2}{dy^2} - \frac{\cosh(y)}{\sinh(y)} \frac{d}{dy} - \frac{\cosh^2\left(\frac{y}{2}\right) + 1}{4\cosh^2\left(\frac{y}{2}\right)}
\end{aligned}$$

Let $\tilde{\mathcal{G}}_{\mathbb{P},\lambda}$, and $\tilde{\mathcal{G}}_{\mathbb{D},\lambda}$ denote the Green's functions of the operators

$$-\lambda^2 + \frac{d^2}{dy^2} + \frac{\cos(y)}{\sin(y)} \frac{d}{dy} \quad \text{and} \quad -\lambda^2 + \frac{d^2}{dy^2} + \frac{\cosh(y)}{\sinh(y)} \frac{d}{dy}$$

on the measure spaces $\frac{\pi \sin(y) dy}{2}$ with $y \in [0, \pi]$ and $\frac{\pi \sinh(y) dy}{2}$ with $y \in [0, \infty)$. They describe the Laplace operators acting on functions [Cha, Chapter VII §5.]. The corresponding

Green's functions have representations analogous to the representation of $\mathcal{G}_{C,\lambda}$ (compare [Da, Chapter 5]). In fact, the substitution $x = -\cos(y)$ transforms the former operator into $-\lambda^2 + (1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$, whose eigenfunctions are the Legendre polynomials [Bat, §10.10.]. We apply a variant of Mehlers integral [Bat, §10.10. (43)]:

$$\begin{aligned}
P_n(-\cos(y)) &= P_n(\cos(\pi - y)) = \frac{1}{\pi} \int_{y-\pi}^{\pi-y} \frac{\exp(\sqrt{-1}(n + \frac{1}{2})x) dx}{\sqrt{2 \cos(x) - 2 \cos(\pi - y)}} \\
&= \frac{1}{\pi} \int_y^{2\pi-y} \frac{\exp(\sqrt{-1}(n + \frac{1}{2})(\pi - x)) dx}{\sqrt{2 \cos(\pi - x) - 2 \cos(\pi - y)}} \\
&= \int_y^\pi \frac{\exp(\sqrt{-1}(n + \frac{1}{2})(\pi - x)) dx}{\pi \sqrt{2 \cos(y) - 2 \cos(x)}} - \int_{-\pi}^{-y} \frac{\exp(\sqrt{-1}(n + \frac{1}{2})(\pi - x)) dx}{\pi \sqrt{2 \cos(y) - 2 \cos(x)}} \\
&= \frac{2(-1)^n}{\pi} \int_y^\pi \frac{\sin((n + \frac{1}{2})(x)) dx}{\sqrt{2 \cos(y) - 2 \cos(x)}}.
\end{aligned}$$

Hence we obtain (compare [Bat, §10.10. (2),(4) and (18)])

$$\begin{aligned}
\tilde{\mathcal{G}}_{\mathbb{P}, \sqrt{\lambda^2 + \frac{1}{4}}}(y) &= -\frac{2}{\pi} \sum_{n=0}^{\infty} P_n(-\cos(y)) \frac{(-1)^n (n + \frac{1}{2})}{\lambda^2 + (n + \frac{1}{2})^2} \\
&= \frac{2}{\pi^2} \int_y^\pi \sum_{n=0}^{\infty} \left(\frac{\sqrt{-1}}{\lambda - \sqrt{-1}(n + \frac{1}{2})} - \frac{\sqrt{-1}}{\lambda + \sqrt{-1}(n + \frac{1}{2})} \right) \frac{\sin((n + \frac{1}{2})x) dx}{\sqrt{2 \cos(y) - 2 \cos(x)}} \\
&= \frac{1}{\pi^2} \int_y^\pi \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\exp(\sqrt{-1}nx)}{\lambda - \sqrt{-1}n} - \frac{\exp(\sqrt{-1}nx)}{\lambda + \sqrt{-1}n} \frac{dx}{\sqrt{2 \cos(y) - 2 \cos(x)}} \\
&= -\frac{2\pi}{\pi^2} \int_y^\pi \frac{\cosh(\lambda(\pi - x)) dx}{\cosh(\lambda\pi) \sqrt{2 \cos(y) - 2 \cos(x)}} \\
&= -\frac{2}{\pi} \int_{|\lambda|y}^{|\lambda|\pi} \frac{\cosh(|\lambda|\pi - x) dx}{\cosh(\lambda\pi) |\lambda| \sqrt{2 \cos(y) - 2 \cos(\frac{x}{\lambda})}}.
\end{aligned}$$

On the other hand, the heat kernel of the hyperbolic plane yields the following representation

$$\begin{aligned}
\tilde{\mathcal{G}}_{\mathbb{D}, \sqrt{\lambda^2 - \frac{1}{4}}}(y) &= -\frac{1}{\pi^{3/2}} \int_0^\infty \int_y^\infty \frac{x \exp\left(-\frac{t}{4} - \frac{x^2}{4t} - \lambda^2 t + \frac{t}{4}\right)}{t^{3/2} \sqrt{2 \cosh(x) - 2 \cosh(y)}} dx dt \\
&= -\frac{2}{\pi} \int_y^\infty \frac{\exp(-|\lambda|x)}{\sqrt{2 \cosh(x) - 2 \cosh(y)}} dx \\
&= -\frac{2}{\pi} \int_{|\lambda|y}^\infty \frac{\exp(-x)}{|\lambda| \sqrt{2 \cosh(\frac{x}{\lambda}) - 2 \cosh(y)}} dx.
\end{aligned}$$

We conclude that the Green's functions $\tilde{\mathcal{G}}_{\mathbb{P}, \lambda}$, and $\tilde{\mathcal{G}}_{\mathbb{D}, \lambda}$ have also the properties (i)–(iv). Due to [Da, Chapter 1.3 and Chapter 1.8] the resolvents of the Laplace operators on the three simply connected Riemann surfaces \mathbb{P}, \mathbb{C} and \mathbb{D} are positivity preserving. Moreover, if \mathbf{H}_0 is an elliptic second order differential operator and V a non-negative potential, the difference of the resolvents

$$(\lambda^2 + \mathbf{H}_0)^{-1} - (\lambda^2 + \mathbf{H}_0 + V)^{-1} = (\lambda^2 + \mathbf{H}_0)^{-1} V (\lambda^2 + \mathbf{H}_0 + V)^{-1}$$

is positivity preserving. Moreover, if in addition $-\frac{d}{dy}(\lambda^2 + \mathbf{H}_0)^{-1}$ is positivity preserving, then also the difference

$$\frac{d}{dy}(\lambda^2 + \mathbf{H}_0 + V)^{-1} - \frac{d}{dy}(\lambda^2 + \mathbf{H}_0)^{-1} = -\frac{d}{dy}(\lambda^2 + \mathbf{H}_0)^{-1} V (\lambda^2 + \mathbf{H}_0 + V)^{-1}$$

is positivity preserving. Hence we may estimate the positive Green's functions

$$\begin{aligned}
0 < -\mathcal{G}_{\mathbb{P}, \lambda}(y) &\leq -\tilde{\mathcal{G}}_{\mathbb{P}, \sqrt{\lambda^2 + \frac{1}{4}}}(y) & 0 < \mathcal{G}'_{\mathbb{P}, \lambda}(y) &\leq \tilde{\mathcal{G}}'_{\mathbb{P}, \sqrt{\lambda^2 + \frac{1}{4}}}(y) \\
0 < -\mathcal{G}_{\mathbb{D}, \lambda}(y) &\leq -\tilde{\mathcal{G}}_{\mathbb{D}, \sqrt{\lambda^2 + \frac{1}{4}}}(y) & 0 < \mathcal{G}'_{\mathbb{D}, \lambda}(y) &\leq \tilde{\mathcal{G}}'_{\mathbb{D}, \sqrt{\lambda^2 + \frac{1}{4}}}(y)
\end{aligned}$$

This proves

Lemma 3.1. *The Green's functions $\mathcal{G}_{\mathbb{P}, \lambda}$, $\mathcal{G}_{\mathbb{D}, \lambda}$ and $\mathcal{G}_{\mathbb{D}, \lambda}$ have the properties*

(i) $0 < -\mathcal{G}_{\cdot, \lambda}(y) \leq \mathbf{O}(1) \exp((\varepsilon - |\lambda|)y)$ with an appropriate $\varepsilon > 0$ and large $|\lambda|y$.

(ii) $0 < \mathcal{G}'_{\cdot, \lambda}(y) \leq \mathbf{O}(1) \exp((\varepsilon - |\lambda|)y)$ with an appropriate $\varepsilon > 0$ and large $|\lambda|y$.

(iii) $0 < -\mathcal{G}_{\cdot, \lambda}(y) \leq -\frac{2}{\pi} \ln(|\lambda|y) + \mathbf{O}(1)$ for small y .

(iv) $0 < \mathcal{G}'_{\cdot, \lambda}(y) \leq \frac{2}{\pi} \frac{1}{y} + \mathbf{O}(1)$ for small y .

q.e.d.

3. Integral kernels of the resolvents of Dirac operators on simply connected Riemann surfaces. The free Dirac operator on \mathbb{C} is translation invariant. Moreover, the free Dirac operators on \mathbb{P} and \mathbb{D} are invariant under group actions of the subgroups $SU(2)$ and $SU(1, 1)$ of the Möbius group, respectively. More precisely, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ and $SU(1, 1)$ the transformation

$$z' = \frac{az + b}{cz + d} \text{ implies } \partial' = (cz + d)^2 \partial, \quad \bar{\partial}' = (\bar{c}\bar{z} + \bar{d})^2 \bar{\partial}, \quad 1 \pm z' \bar{z}' = \frac{1 \pm z \bar{z}}{|cz + d|^2} \text{ and}$$

$$\begin{pmatrix} \sqrt{-1}\lambda & -(1 \pm z' \bar{z}') \partial' \\ (1 \pm z' \bar{z}') \bar{\partial}' & \sqrt{-1}\lambda \end{pmatrix} = \begin{pmatrix} cz + d & 0 \\ 0 & \bar{c}\bar{z} + \bar{d} \end{pmatrix} \begin{pmatrix} \sqrt{-1}\lambda & -(1 \pm z \bar{z}) \partial \\ (1 \pm z \bar{z}) \bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \begin{pmatrix} cz + d & 0 \\ 0 & \bar{c}\bar{z} + \bar{d} \end{pmatrix}^{-1},$$

$$\begin{pmatrix} \sqrt{-1}\lambda & (1 \pm z' \bar{z}') \partial' \\ -(1 \pm z' \bar{z}') \bar{\partial}' & \sqrt{-1}\lambda \end{pmatrix} = \begin{pmatrix} cz + d & 0 \\ 0 & \bar{c}\bar{z} + \bar{d} \end{pmatrix} \begin{pmatrix} \sqrt{-1}\lambda & (1 \pm z \bar{z}) \partial \\ -(1 \pm z \bar{z}) \bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \begin{pmatrix} cz + d & 0 \\ 0 & \bar{c}\bar{z} + \bar{d} \end{pmatrix}^{-1},$$

respectively. Therefore the spin bundle of the compact Riemann surface \mathbb{P} is the trivial \mathbb{C}^2 -bundle on the two members of the covering $\mathbb{P} = \{z \in \mathbb{C}\} \cup \{z' \in \mathbb{C}\}$ with the transformation $z' = -1/z$ and the transition matrix $\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$, which transforms the spinors on $\{z \in \mathbb{C}\}$ into the spinors on $\{z' \in \mathbb{C}\}$. The spin bundles of \mathbb{C} and \mathbb{D} are the trivial \mathbb{C}^2 -bundles over these non-compact Riemann surfaces. The translation invariance of the free Dirac operator on \mathbb{C} implies that the resolvent $R_{\mathbb{C}}(0, 0, \sqrt{-1}\lambda) = \left(\frac{\sqrt{-1}\lambda}{\bar{\partial}} \frac{-\partial}{\sqrt{-1}\lambda} \right)^{-1}$ has the integral kernel $\mathcal{K}_{\mathbb{C}, \lambda}(z, z') \frac{d\bar{z}' \wedge dz'}{2\sqrt{-1}}$ with

$$\mathcal{K}_{\mathbb{C}, \lambda}(z, z') = \begin{pmatrix} \sqrt{-1}\lambda & \partial \\ -\bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \begin{pmatrix} \mathcal{G}_{\mathbb{C}, \lambda}(2|z - z'|) & 0 \\ 0 & \mathcal{G}_{\mathbb{C}, \lambda}(2|z - z'|) \end{pmatrix} \frac{d\bar{z}' \wedge dz'}{2\sqrt{-1}}.$$

On \mathbb{P} and \mathbb{D} we use the invariance under $SU(2)$ and $SU(1, 1)$. The transformed coordinate under the Möbius transformation $\begin{pmatrix} 1 & -z' \\ \pm \bar{z}' & 1 \end{pmatrix} / \sqrt{1 \pm z' \bar{z}'} \in SU(2)$ and $SU(1, 1)$ vanishes at $z' \in \mathbb{P}$ and \mathbb{D} , respectively. Therefore the integral kernels of the resolvents $\left(\frac{\sqrt{-1}\lambda}{(1 \pm z \bar{z}) \bar{\partial}} \frac{-(1 \pm z \bar{z}) \partial}{\sqrt{-1}\lambda} \right)^{-1}$ on \mathbb{P} and \mathbb{D} have the integral kernels $\mathcal{K}_{\mathbb{P}, \lambda}(z, z') \frac{d\bar{z}' \wedge dz'}{2\sqrt{-1}(1+z' \bar{z}')^2}$ and $\mathcal{K}_{\mathbb{D}, \lambda}(z, z') \frac{d\bar{z}' \wedge dz'}{2\sqrt{-1}(1-z' \bar{z}')^2}$ with

$$\begin{aligned} \mathcal{K}_{\mathbb{P}, \lambda}(z, z') &= \begin{pmatrix} \sqrt{-1}\lambda & (1 + z \bar{z}) \partial \\ -(1 + z \bar{z}) \bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \begin{pmatrix} \frac{1+z' \bar{z}'}{1+z \bar{z}'} \mathcal{G}_{\mathbb{P}, \lambda}(2d_{\mathbb{P}}(z, z')) & 0 \\ 0 & \frac{1+z' \bar{z}'}{1+\bar{z} \bar{z}'} \mathcal{G}_{\mathbb{P}, \lambda}(2d_{\mathbb{P}}(z, z')) \end{pmatrix} \\ \mathcal{K}_{\mathbb{D}, \lambda}(z, z') &= \begin{pmatrix} \sqrt{-1}\lambda & (1 - z \bar{z}) \partial \\ -(1 - z \bar{z}) \bar{\partial} & \sqrt{-1}\lambda \end{pmatrix} \begin{pmatrix} \frac{1-z' \bar{z}'}{1-z \bar{z}'} \mathcal{G}_{\mathbb{D}, \lambda}(2d_{\mathbb{D}}(z, z')) & 0 \\ 0 & \frac{1-z' \bar{z}'}{1-\bar{z} \bar{z}'} \mathcal{G}_{\mathbb{D}, \lambda}(2d_{\mathbb{D}}(z, z')) \end{pmatrix}. \end{aligned}$$

Here $d_{\mathbb{P}}(z, z')$ and $d_{\mathbb{D}}(z, z')$ denote the distance between z and z' with respect to the invariant metrics $\frac{dzd\bar{z}}{(1\pm z\bar{z})^2}$ on \mathbb{P} and \mathbb{D} , respectively.

4. Integral kernels of the resolvents of Dirac operators on compact Riemann surfaces. A spin bundle of the elliptic Riemann surface \mathbb{C}/Λ is the trivial \mathbb{C}^2 bundle. Finally, a spin bundle of the hyperbolic compact Riemann surface \mathbb{D}/Γ is the induced bundle of the discrete Fuchsian subgroup $\Gamma \subset SU(1, 1)$ of the following action on the sections of the trivial spin bundle on \mathbb{D} :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts on spinors as } \psi \mapsto \psi' \text{ with } \psi'(z) = \begin{pmatrix} a - cz & 0 \\ 0 & \bar{a} - \bar{c}\bar{z} \end{pmatrix}^{-1} \psi \left(\frac{dz - b}{a - cz} \right).$$

Consequently, the resolvent $R_{\mathbb{C}/\Lambda}(0, 0, \sqrt{-1}\lambda) = \left(\begin{smallmatrix} \sqrt{-1}\lambda & -\partial \\ \bar{\partial} & \sqrt{-1}\lambda \end{smallmatrix} \right)^{-1}$ of the free Dirac operator on \mathbb{C}/Λ has the integral kernel

$$\sum_{\gamma \in \Lambda} \mathcal{K}_{\mathbb{C}, \lambda}(z, z' + \gamma) \frac{d\bar{z}' \wedge dz'}{2\sqrt{-1}}$$

with $z, z' \in \Delta$. Due to property (i) of Lemma 3.1 these sum converges for all non-vanishing real λ . Analogously, the resolvent $R_{\mathbb{D}/\Gamma}(0, 0, \sqrt{-1}\lambda) = \left(\begin{smallmatrix} \sqrt{-1}\lambda & -(1-z\bar{z})\partial \\ (1-z\bar{z})\bar{\partial} & \sqrt{-1}\lambda \end{smallmatrix} \right)^{-1}$ of the free Dirac operator on \mathbb{D}/Γ has the integral kernel

$$\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma} \mathcal{K}_{\mathbb{D}, \lambda} \left(z, \frac{dz' - c}{a - cz'} \right) \begin{pmatrix} a - cz' & 0 \\ 0 & \bar{a} - \bar{c}\bar{z}' \end{pmatrix} \frac{d\bar{z}' \wedge dz'}{2\sqrt{-1}(1 - z'\bar{z}')^2}.$$

5. Banach spaces of spinors The L^p -spinors on \mathbb{C} belong to $L^p(\mathbb{C}, \mathbb{H})$. Moreover, on \mathbb{P} and \mathbb{D} the L^p -spinors have finite norms

$$\|f\| = \left(\int_{\mathbb{P}} |f(z)|^p (1 + z\bar{z})^{\left(\frac{p}{2}-2\right)} \frac{d\bar{z} \wedge dz}{2\sqrt{-1}} \right)^{\frac{1}{p}} \quad \|f\| = \left(\int_{\mathbb{D}} |f(z)|^p (1 - z\bar{z})^{\left(\frac{p}{2}-2\right)} \frac{d\bar{z} \wedge dz}{2\sqrt{-1}} \right)^{\frac{1}{p}},$$

which are invariant under the actions of $SU(2)$ and $SU(1, 1)$. Moreover, the L^p -spinors on \mathbb{C}/Λ are defined as sections of the spin bundle on \mathbb{C}/Λ , with finite norm

$$\|f\| = \left(\int_{\Delta} |f|^p \frac{d\bar{z} \wedge dz}{2\sqrt{-1}} \right)^{\frac{1}{p}}.$$

Finally, the L^p -spinors on \mathbb{D}/Γ are defined as section of the spin bundle on \mathbb{D}/Γ , with finite norm

$$\|f\| = \left(\int_{\Delta} |f|^p (1 - z\bar{z})^{\frac{p}{2}} \frac{d\bar{z} \wedge dz}{2\sqrt{-1}(1 - z\bar{z})^2} \right)^{\frac{1}{p}}.$$

Let $R_{\mathbb{P}}(V, W, \lambda)$ and $R_{\mathbb{D}}(V, W, \lambda)$ denote the resolvents

$$\begin{pmatrix} \lambda - V & -(1 + z\bar{z})\partial \\ (1 + z\bar{z})\bar{\partial} & \lambda - W \end{pmatrix}^{-1} \quad \begin{pmatrix} \lambda - V & -(1 - z\bar{z})\partial \\ (1 - z\bar{z})\bar{\partial} & \lambda - W \end{pmatrix}^{-1}$$

considered as operators from the L^p -spinors into the L^q -spinors. The corresponding free resolvents $R_{\mathbb{P}}(0, 0, \sqrt{-1}\lambda)$ and $R_{\mathbb{D}}(0, 0, \sqrt{-1}\lambda)$ have the integral kernels

$$K_{\mathbb{P},\lambda}(z, z') \frac{d\bar{z} \wedge dz}{2\sqrt{-1}(1 + z'\bar{z}')^2} \quad K_{\mathbb{D},\lambda}(z, z') \frac{d\bar{z} \wedge dz}{2\sqrt{-1}(1 - z'\bar{z}')^2}.$$

Analogously let $R_{\mathbb{C}/\Lambda}(V, W, \lambda)$ and $R_{\mathbb{D}/\Gamma}(V, W, \lambda)$ denote the resolvents of Dirac operators on with potentials V and W on the compact Riemann surfaces \mathbb{C}/Λ and \mathbb{D}/Γ . Therefore for all compact Riemann surfaces $X = \mathbb{P}, \mathbb{C}/\Lambda, \mathbb{D}/\Gamma$ the resolvents of the Dirac operators with potentials V and W are equal to

$$\begin{aligned} R_X(V, W, \sqrt{-1}\lambda) &= \left(\mathbf{1} - R_X(0, 0, \sqrt{-1}\lambda) \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \right)^{-1} R_X(0, 0, \sqrt{-1}\lambda) \\ &= R_X(0, 0, \sqrt{-1}\lambda) \left(\mathbf{1} - \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} R_X(0, 0, \sqrt{-1}\lambda) \right)^{-1}. \end{aligned}$$

6. The resolvents of Dirac operators with L^2 -potentials on compact Riemann surfaces.

Theorem 3.2. *For all $1 \leq p, q < \infty$ with $\frac{1}{p} < \frac{1}{q} + \frac{1}{2}$, there exists a constant $C_p > 0$, with the following property: For all compact Riemann surfaces X and all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all real $\lambda \in (-\infty, -\delta) \cup (\delta, \infty)$ the mapping $(V, W) \mapsto R_X(V, W, \sqrt{-1}\lambda)$ is holomorphic and weakly continuous from the weakly compact space of all potentials, whose restrictions to all ε -balls of X (with respect to the elliptic, flat or hyperbolic metric) have L^2 -norm not greater than C_p (with respect to the induced measure), into the compact operators from the L^p -spinors into the L^q -spinors.*

Proof. Due to [St, Chapter V. §.3.4 Lemma 3.] the operators $\partial (\mathbf{1} - \partial\bar{\partial})^{-\frac{1}{2}}$ and $\bar{\partial} (\mathbf{1} - \partial\bar{\partial})^{-\frac{1}{2}}$ are bounded operators on $L^p(\mathbb{C})$ with $1 < p < 2$. Therefore any function $f \in L^p(\mathbb{C})$ with either $\partial f \in L^p(\mathbb{C})$ or $\bar{\partial} f \in L^p(\mathbb{C})$ belongs to the Sobolev space $W^{1,p}(\mathbb{C})$ [St, Chapter V. §.3.4 Theorem 3.]. Since \mathbb{C} and \mathbb{D} are homogeneous spaces, they obey the assumptions of the Sobolev Embedding [Au, Theorem 2.21] on Riemannian manifolds. We conclude that for all $1 < p < 2$ the resolvent $R_X(0, 0, \sqrt{-1}\lambda)$ considered as an operator from the space of L^p -spinors into the space of $L^{\frac{2p}{2-p}}$ -spinors are bounded. Moreover, due to Lemma 3.1 there exists a constant $C_p > S_p^{-1}$ such that $\|R_X(0, 0, \sqrt{-1}\lambda)\| \leq S_p$ for all $\lambda \in (-\infty, -1) \cup (1, \infty)$. Now we decompose this resolvent into the sum

$$R_X(0, 0, \sqrt{-1}\lambda) = R_{X,\varepsilon'\text{-near}}(0, 0, \sqrt{-1}\lambda) + R_{X,\varepsilon'\text{-distant}}(0, 0, \sqrt{-1}\lambda),$$

whose integral kernel either vanish or are equal to the integral kernel of $R_X(0, 0, \sqrt{-1}\lambda)$, in cases that z and z' have distance larger than ε or smaller than ε and vice versa, respectively. Obviously the norm of the first term is smaller than S_p . If the potentials V

and W belong to the set described in the Lemma, the operator $\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} R_{X, \varepsilon' \text{-near}}(0, 0, \sqrt{-1}\lambda)$ has for small ε' norm smaller than $\left(\frac{\text{vol}(B(0, \varepsilon + \varepsilon'))}{\text{vol}(B(0, \varepsilon))}\right)^{1/p}$. In fact, for all $x \in X$ the restriction of $R_{X, \varepsilon' \text{-near}}(0, 0, \sqrt{-1}\lambda)\psi$ to $B(x, \varepsilon)$ is smaller than the norm of the restriction of ψ to $B(x, \varepsilon + \varepsilon')$. Since X is either the homogeneous space \mathbb{P} or a quotient of the homogeneous spaces \mathbb{C} or \mathbb{D} by a discrete group, for all small ε and all L^p -functions on X , the L^p -norm of the function

$$x \mapsto \|f|_{B(x, \varepsilon)}\|_p$$

is equal to $\text{vol}^{\frac{1}{p}}(B(0, \varepsilon))$ times the L^p -norm $\|f\|_p$ of f . We conclude that for small ε' the operator $\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} R_{X, \varepsilon' \text{-near}}(0, 0, \sqrt{-1}\lambda)$ has norm smaller than 1. Due to Lemma 3.1 (ii), in the limit $|\lambda| \rightarrow \infty$ the norm of the second term converge to zero $\lim_{|\lambda| \rightarrow \infty} \|R_{X, \varepsilon' \text{-distant}}(0, 0, \sqrt{-1}\lambda)\| = 0$. Hence there exists a $\delta > 0$, such that the operator $\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} R_X(0, 0, \sqrt{-1}\lambda)$ on the L^p -spinors has for all $\lambda \in (-\infty, -\delta) \cup (\delta, \infty)$ norm smaller than 1. Consequently the von Neumann series

$$R_X(V, W, \sqrt{-1}\lambda) = \sum_{l=0}^{\infty} R_X(V, W, \sqrt{-1}\lambda) \left(\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} R_X(0, 0, \sqrt{-1}\lambda)\right)^l$$

converges to a holomorphic function with values in the operators from the L^p -spinors into the $L^{\frac{2p}{2-p}}$ -spinors. Moreover, due to Kondrakov's Theorem [Au, Theorem 2.34] the resolvent $R_X(0, 0, \sqrt{-1}\lambda)$ considered as an operator from the L^p -spinors into the L^q -spinors with $1 \leq q < \frac{2p}{2-p}$ is compact. Due to [L-T, Theorem II.5.11], all Banach spaces of L^q -spinors have a *Schauder basis*. Consequently they have the *approximation property*, and all compact operators into one of these Banach spaces of L^q -spinors are norm-limits of finite rank operators (compare [L-T, Section I.1.a]). Hence all terms in the von Neumann series are norm-limits of weakly continuous functions from the set described in the Lemma into the compact operators from the L^p -spinors into the L^q spinors. Since this set is weakly compact, the uniform limit of weakly continuous functions is again a weakly continuous function [R-S-I, Theorem IV.8]. **q.e.d.**

We shall explain the relation of these Dirac operators and holomorphic structures. In the introduction we mentioned already, that Dirac operators on \mathbb{C} are the composition of holomorphic structures with an invertible operator

$$\begin{pmatrix} U & \partial \\ -\bar{\partial} & \bar{U} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}.$$

The Dirac operator on \mathbb{P} and \mathbb{D} with potentials $(1 \pm z\bar{z})U$ and $(1 \pm z\bar{z})\bar{U}$ are equal to

$$\begin{pmatrix} (1 \pm z\bar{z})U & (1 \pm z\bar{z})\partial \\ -(1 \pm z\bar{z})\bar{\partial} & (1 \pm z\bar{z})\bar{U} \end{pmatrix} = (1 \pm z\bar{z}) \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}.$$

Hence these Dirac operators are also compositions of holomorphic structures with invertible operators. We remark that the L^2 -norms of the potentials $(1 \pm z\bar{z})U$ with respect to the

induced measures $\frac{d\bar{z}\wedge dz}{2\sqrt{-1}(1\pm z\bar{z})^2}$ coincides with the integrals over $\frac{1}{2\sqrt{-1}}Q \wedge \bar{Q}$ with the corresponding Hopf fields $Q = -\bar{U}d\bar{z}$.

Finally let us deduce a simple criterion for a bounded sequence of square integrable Hopf fields on a compact Riemann surface X , whether they contain subsequences in the sets of the form described in Theorem 3.2 or not. Due to the Banach–Alaoglu theorem [R-S-I, Theorem IV.21] and the Riesz Representation theorem [Ro, Chapter 13 Section 5] any bounded sequence $(Q_n)_{n \in \mathbb{N}}$ of square integrable Hopf fields has a subsequence, with the property that the corresponding sequence of measures $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$ converge weakly to a finite Baire measure on X . This subsequence is contained in a set of the form described in Theorem 3.2, if the limit of the measure does not contain point measures of mass larger or equal to the constant S_p^{-2} . In fact, if the weak limit of the measures does not contain point measures of mass larger or equal to S_p^{-2} , we may cover X by open sets, whose measures with respect to the limit of the measures is smaller than S_p^{-2} . Due to the compactness of X this open covering has a finite subcovering. For any finite open covering, the function on X , which associates to each x the radius of the maximal open disk around x , which is entirely contained in one member of the covering, is continuous. We exclude the trivial case, where one member of the covering contains the whole of X and therefore all disks. So this function is the maximum of the distances of the corresponding point to all complements of the members of the covering. Therefore, there exists a small $\varepsilon > 0$, such that all disks with radius 2ε are contained in one member of the finite subcovering. Obviously, for any member of the subcovering there exists a continuous $[0, 1]$ -valued function, whose support is contained in this member of the subcovering, and which is equal to 1 on those disks $B(x, \varepsilon)$, whose extensions $B(x, 2\varepsilon)$ are contained in this member of the subcovering. Since the sequence of measures converges weakly, the integrals of these functions with respect to the measures $\frac{1}{2\sqrt{-1}}Q_n(x) \wedge \bar{Q}_n(x)$ corresponding to the sequence are also smaller than S_p^{-2} , with the exception of finitely many elements of the sequence. This shows that with the exception of finitely many elements of the sequence, the L^2 -norms of the restrictions of Q_n to all ε -balls are smaller than some $C_p < S_p^{-1}$.

Lemma 3.3. *If a weak limit of the sequence of finite Baire measures $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n d^2x$ on the compact Riemann surface X does not contain point measures with mass larger or equal to S_p^{-2} , then there exists a $C_p < S_p^{-1}$, an $\varepsilon > 0$ and a subsequence of the bounded sequence Q_n of square integrable Hopf fields, whose L^2 -norms of the restrictions of Q_n to all ε -balls is smaller than C_p .* **q.e.d.**

4 The Riemann–Roch Theorem

In this section we shall prove that all holomorphic quaternionic line bundles with square integrable Hopf fields obey Serre Duality and the Riemann–Roch Theorem. In general, the holomorphic sections of a holomorphic quaternionic line bundle with square integrable Hopf field are not continuous. Therefore we cannot use a non-trivial meromorphic section in order to determine the Chern class of the bundle (compare [F-L-P-P, §2.3]). Hence we use sheaf theory.

Let U be a potential in $L^2_{\text{loc}}(\Omega)$ over an open subset $\Omega \subset \mathbb{C}$. For all $1 < p < 2$ the operator $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ defines a linear operator from $W^{1,p}_{\text{loc}}(\Omega, \mathbb{H})$ onto $L^p_{\text{loc}}(\Omega, \mathbb{H})$. Due to [St, Chapter V. §3.4 Lemma 3.] the operators $\partial(\mathbf{1} - \partial\bar{\partial})^{-\frac{1}{2}}$ and $\bar{\partial}(\mathbf{1} - \partial\bar{\partial})^{-\frac{1}{2}}$ are bounded operators on $L^p(\mathbb{C})$ with $1 < p < 2$. Therefore the operator $\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}$ defines an isomorphism from $W^{1,p}(\Omega, \mathbb{H})$ onto $L^p(\Omega, \mathbb{H})$. For any holomorphic quaternionic line bundle on a Riemann surface X , whose holomorphic complex line bundle corresponds to \mathcal{O}_D , let \mathcal{Q}_D denote the sheaf of holomorphic sections, and $\mathcal{W}_D^{1,p}$ the corresponding sheaf of $W^{1,p}_{\text{loc}}$ -sections. Moreover, let \mathcal{L}_{D-K}^p denote the sheaf of L^p_{loc} -sections of the quaternionic line bundle corresponding to \mathcal{Q}_D tensored with the inverse of the canonical line bundle. If $g dz d\bar{z}$ denotes a hermitian metric with respect to local coordinates z on the compact Riemann surface X , then the local operators

$$\frac{1}{g} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$$

fit together to a global operator from $H^0(X, \mathcal{W}_D^{1,p})$ into $H^0(X, \mathcal{L}_{D-K}^p)$. In fact, under the transformation $z \mapsto z'$ this operator transforms to

$$\frac{1}{g'} \begin{pmatrix} \bar{\partial}' & -\bar{U}' \\ U' & \partial' \end{pmatrix} = \frac{1}{g} \left| \frac{dz'}{dz} \right|^2 \begin{pmatrix} \frac{\bar{dz}}{dz'} & 0 \\ 0 & \frac{dz}{dz'} \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} = \frac{1}{g} \begin{pmatrix} \frac{dz'}{dz} & 0 \\ 0 & \frac{dz'}{dz} \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}.$$

Moreover, the holomorphic cocycle of the underlying holomorphic complex line bundle does only change the Hopf field $Q = -\bar{U} d\bar{z}$. Consequently, the holomorphic structure of the quaternionic line bundle, which is locally given by operators of the form $\frac{1}{g} \begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$, induces a morphism $\mathcal{W}_D^{1,p} \rightarrow \mathcal{L}_{D-K}^p$ which fits to the following exact sequence of sheaves [F-L-P-P, §2.2]

$$0 \rightarrow \mathcal{Q}_D \hookrightarrow \mathcal{W}_D^{1,p} \rightarrow \mathcal{L}_{D-K}^p \rightarrow 0.$$

Standard arguments [Fo, Theorem 12.6.] show that the first cohomology group of the sheaf $\mathcal{W}_D^{1,p}$ vanish. Consequently, the corresponding long exact cohomology sequence [Fo, §15.] shows that the cokernel of the holomorphic structure, considered as a Fredholm operator from $H^0(X, \mathcal{W}_D^{1,p})$ into $H^0(X, \mathcal{L}_{D-K}^p)$ is naturally isomorphic to the first cohomology group $H^1(X, \mathcal{Q}_D)$ of the sheaf \mathcal{Q}_D . On the other hand, this cokernel is dual to the kernel of the transposed operator acting on the dual space of $H^0(X, \mathcal{L}_{D-K}^p)$, which is equal to $H^0(X, \mathcal{L}_{K-D}^{\frac{p}{p-1}})$. This transposed operator defines a natural holomorphic structure on the quaternionic line bundle over \mathcal{O}_{K-D} [F-L-P-P, §2.3.]. The corresponding sheaf of holomorphic sections is denoted by \mathcal{Q}_{K-D} . Hence we have proven (compare [Na, §8.–§9.])

Sérre Duality 4.1. *Let X be a compact Riemann surface and \mathcal{Q}_D the sheaf of holomorphic sections of a holomorphic structure with a square integrable Hopf field Q (i. e. $\frac{1}{2\sqrt{-1}} \int_X Q \wedge \bar{Q} < \infty$) on the quaternionic line bundle over the complex line bundle corresponding to \mathcal{O}_D . Then the Čech cohomology groups $H^1(X, \mathcal{Q}_D)$ and $H^0(X, \mathcal{Q}_{K-D})$ are naturally dual to each other. **q.e.d.***

Riemann–Roch Theorem 4.2. *Let X be a compact Riemann surface and \mathcal{Q}_D the sheaf of holomorphic sections of a holomorphic quaternionic line bundle with square integrable Hopf fields (i. e. $\frac{1}{2\sqrt{-1}} \int_X Q \wedge \bar{Q} < \infty$). over the complex line bundle corresponding to \mathcal{O}_D . Then the quaternionic dimensions of the corresponding Čech cohomology groups are finite and obey the formula*

$$\dim_{\mathbb{Q}} H^0(X, \mathcal{Q}_D) - \dim_{\mathbb{Q}} H^1(X, \mathcal{Q}_D) = 1 - g + \deg(D).$$

Proof. Due to the long exact cohomology sequence corresponding to the exact sequence of sheaves [Fo, §15.]

$$0 \rightarrow \mathcal{Q}_D \rightarrow \mathcal{Q}_{D'} \rightarrow \mathcal{Q}_{D'}/\mathcal{Q}_D \rightarrow 0$$

and Lemma 2.7 the *Riemann–Roch Theorem* for the sheaf \mathcal{Q}_D is equivalent to the *Riemann–Roch Theorem* for the sheaf $\mathcal{Q}_{D'}$ with $D \leq D'$. Since for all pairs of divisors D and D' there exists a divisor D'' with $D \leq D''$ and $D' \leq D''$, this equivalence holds also for arbitrary D and D' . Consequently, it suffices to prove the *Riemann–Roch Theorem* for the holomorphic quaternionic line bundles with one fixed underlying holomorphic complex line bundle. Theorem 3.2 shows that holomorphic structures with square integrable Hopf fields on the spin bundle, considered as a quaternionic line bundle, are Fredholm operators of index zero from $H^0(X, \mathcal{W}_D^{1,p})$ into $H^0(X, \mathcal{L}_D^p)$, where D is the corresponding square root of the canonical divisor, i. e. $2D = K$. This implies $\deg(D) = g - 1$ and the claim follows from the proof of *Sérre Duality* 4.1. **q.e.d.**

5 A Bäcklund transformation

In the following discussion concerning this transformation we make use of the *Lorentz spaces* $L^{p,q}$. These rearrangement invariant Banach spaces are an extension of the family of the usual Banach spaces L^p indexed by an additional parameter $1 \leq q \leq \infty$ for $1 < p < \infty$. For $p = 1$ or $p = \infty$ we consider only the *Lorentz spaces* $L^{p,\infty}$, which in these cases are isomorphic to L^p ([S-W, Chapter V. §3.], [B-S, Chapter 4 Section 4.] and [Zi, Chapter 1. Section 8.]). We recall some properties of these Banach spaces:

- (i) For $1 < p \leq \infty$ the *Lorentz spaces* $L^{p,p}$ coincide with the usual L^p -spaces. Moreover, the *Lorentz space* $L^{1,\infty}$ coincides with the usual Banach space L^1 .
- (ii) On a finite measure space the *Lorentz space* $L^{p,q}$ is contained in $L^{p',q'}$ either if $p > p'$ or if $p = p'$ and $q \leq q'$.

In [O] Hölder's inequality and Young's inequality are generalized to these *Lorentz spaces* ([B-S, Chapter 4 Section 7.] and [Zi, Chapter 2. Section 10.]):

Generalized Hölder's inequality 5.1. *Either for $1/p_1 + 1/p_2 = 1/p_3 < 1$ and $1/q_1 + 1/q_2 \geq 1/q_3$ or for $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 \geq 1$ and $(p_3, q_3) = (1, \infty)$ there exists some constant $C > 0$ with*

$$\|fg\|_{(p_3, q_3)} \leq C \|f\|_{(p_1, q_1)} \|g\|_{(p_2, q_2)}.$$

Generalized Young's inequality 5.2. *Either for $1/p_1 + 1/p_2 - 1 = 1/p_3 > 0$ and $1/q_1 + 1/q_2 \geq 1/q_3$ or for $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 \geq 1$ and $(p_3, q_3) = (\infty, \infty)$ there exists some constant $C > 0$ with*

$$\|f * g\|_{(p_3, q_3)} \leq C \|f\|_{(p_1, q_1)} \|g\|_{(p_2, q_2)}.$$

Therefore, the resolvent of the Dirac operators on \mathbb{C} is a bounded operators from the L^1 -spinors into the $L^{2,\infty}$ -spinors, from the $L^{2,1}$ -spinors into the continuous spinors, from the L^p -spinors into the $L^{q,p}$ -spinors, and finally from the $L^{p,q}$ -spinors into the L^q -spinors, with $1 < p < 2$ and $q = 2p/(2-p)$. Moreover, the Sobolev constant S_p (compare with Lemma 3.1 and Theorem 3.2) is equal to the corresponding norm $\|f\|_{2,\infty}$ times the corresponding constant of the *Generalized Young's inequality 5.2*.

Let ξ and χ be two elements in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \partial \end{pmatrix}$ on an open domain $\Omega \subset \mathbb{C}$. If χ does not vanish, then the quotient of these two holomorphic sections of the corresponding holomorphic quaternionic line bundle is equal to

$$\begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} = \frac{1}{\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2} \begin{pmatrix} \bar{\chi}_1 & \bar{\chi}_2 \\ -\chi_2 & \chi_1 \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}.$$

The derivatives of these quaternionic-valued functions are equal to

$$\begin{aligned} d \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} &= \left(\begin{pmatrix} d\bar{z} & 0 \\ 0 & dz \end{pmatrix} \begin{pmatrix} 0 & \bar{A} \\ -A & 0 \end{pmatrix} + \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} \partial & 0 \\ 0 & \bar{\partial} \end{pmatrix} \right) \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} \\ d \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix} &= \left(\begin{pmatrix} d\bar{z} & 0 \\ 0 & dz \end{pmatrix} \begin{pmatrix} 0 & \bar{A} \\ -A & 0 \end{pmatrix} - \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} B & \bar{U} \\ -U & \bar{B} \end{pmatrix} \right) \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}, \text{ with} \\ \begin{pmatrix} B & \bar{U} \\ -U & \bar{B} \end{pmatrix} &= - \begin{pmatrix} \partial \chi_1 & -\partial \bar{\chi}_2 \\ \bar{\partial} \chi_2 & \bar{\partial} \bar{\chi}_1 \end{pmatrix} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1}. \end{aligned}$$

Therefore, the derivative of the foregoing quotient is equal to

$$\begin{aligned} d \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} &= \\ &= \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \left(d \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} - \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix} \right) \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} \\ &= \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \left(\begin{pmatrix} \partial & 0 \\ 0 & \bar{\partial} \end{pmatrix} + \begin{pmatrix} B & \bar{U} \\ -U & \bar{B} \end{pmatrix} \right) \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}. \end{aligned}$$

The non-vanishing section $\begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}$ induces a flat connection on the quaternionic line bundle. The zero curvature equation takes the form

$$\left[\begin{pmatrix} \partial + B & \bar{U} \\ A & \partial \end{pmatrix}, \begin{pmatrix} \bar{\partial} & -\bar{A} \\ -U & \bar{\partial} + \bar{B} \end{pmatrix} \right] = 0.$$

In the framework of 'quaternionic function theory' [F-L-P-P] this equation takes the form

$$\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \begin{pmatrix} \partial + B & \bar{U} \\ -U & \bar{\partial} + \bar{B} \end{pmatrix} = \begin{pmatrix} \partial + B & \bar{A} \\ -A & \bar{\partial} + \bar{B} \end{pmatrix} \begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \partial \end{pmatrix}.$$

This implies the equation

$$\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix} \begin{pmatrix} \partial + B & \bar{U} \\ -U & \bar{\partial} + \bar{B} \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} = 0.$$

Therefore the quaternionic-valued function $\begin{pmatrix} \psi_1 & \bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \partial + B & \bar{U} \\ -U & \bar{\partial} + \bar{B} \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$. On the other hand the quaternionic-valued function $\begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1}$ obeys the differential equation

$$\begin{aligned} \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1} &= \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} \bar{\chi}_1 & -\chi_2 \\ \bar{\chi}_2 & \chi_1 \end{pmatrix} \frac{1}{\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2} \\ &= \left(\begin{pmatrix} -\bar{B} & -U \\ \bar{U} & -B \end{pmatrix} - \begin{pmatrix} \bar{\partial} \ln(\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2) & 0 \\ 0 & \partial \ln(\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2) \end{pmatrix} \right) \begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & -U \\ \bar{U} & 0 \end{pmatrix} \begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1}. \end{aligned}$$

Here we used

$$\bar{\partial} \ln(\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2) = \frac{\chi_1 \bar{\partial} \bar{\chi}_1 + \bar{\chi}_2 \bar{\partial} \chi_2}{\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2} = -\bar{B} \quad \partial \ln(\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2) = \frac{\bar{\chi}_1 \partial \chi_1 + \chi_2 \partial \bar{\chi}_2}{\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2} = -B.$$

Therefore this function belongs to the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$.

Bäcklund transformation 5.3. *Let $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ and $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ be two spinors in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \partial \end{pmatrix}$ on an open domain $\Omega \subset \mathbb{C}$ with square integrable potential $A \in L^2_{\text{loc}}(\Omega)$. Moreover, let χ have no zeroes on Ω (in the sense of Order of zeroes 2.6). Then the components of the quaternionic-valued functions $\begin{pmatrix} B & \bar{U} \\ -U & \bar{B} \end{pmatrix} = - \begin{pmatrix} \partial \chi_1 & -\partial \bar{\chi}_2 \\ \bar{\partial} \chi_2 & \bar{\partial} \bar{\chi}_1 \end{pmatrix} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1}$ belong to $U \in L^2_{\text{loc}}(\Omega)$ and $B \in L^{2,\infty}_{\text{loc}}(\Omega)$. More precisely, the function $\bar{\partial} B$ is a measure on Ω without point measures. Moreover, the derivative of the quotient $\begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$ is equal to*

$$d \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} = \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} \partial + B & \bar{U} \\ -U & \bar{\partial} + \bar{B} \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}.$$

Furthermore, the function $\begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \partial + B & \bar{U} \\ -U & \bar{\partial} + \bar{B} \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and the function $\begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix} = \begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1}$ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$. In particular, the quotient $\begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$ belongs to $\bigcap_{1 < p < 2} W^{2,p}_{\text{loc}}(\Omega, \mathbb{H})$.

Conversely, if $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ belong on Ω to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ to the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$, then

$$d \begin{pmatrix} f_1 & -\bar{f}_2 \\ f_2 & \bar{f}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix} \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

is a closed quaternionic-valued form on Ω . If in addition ϕ has no zeroes on Ω , then the two spinors $\begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix}^{-1}$ and $\begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix}^{-1} \begin{pmatrix} f_1 & -\bar{f}_2 \\ f_2 & \bar{f}_1 \end{pmatrix}$ belong on Ω to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \partial \end{pmatrix}$ with potential $A = \frac{\bar{\phi}_1 \partial \phi_2 - \phi_2 \bar{\partial} \bar{\phi}_1}{\phi_1 \phi_1 + \phi_2 \phi_2} \in L^2_{\text{loc}}(\Omega)$.

Proof. This proposition establishes a one-to-one correspondence between two holomorphic sections ξ and χ of a holomorphic quaternionic line bundle, and two holomorphic sections ψ and ϕ of two paired quaternionic holomorphic line bundles. We prove this proposition in four steps. In steps 1–3 we proof the statements concerning the mapping from two spinors χ and ξ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \partial \end{pmatrix}$ to two ‘paired’ spinors ψ and ϕ . In the final step we prove the statements concerning the inverse transformation from two ‘paired’ spinors ϕ and ψ to two ‘holomorphic’ spinors ξ and χ of one holomorphic quaternionic bundle.

1. For potentials $A \in L^2_{\text{loc}}(\Omega)$. If the Hopf field belongs locally to $L^{2,1}(\Omega)$, then the *Generalized Hölder’s inequality* 5.1 and *Generalized Young’s inequality* 5.2 together with the arguments in section 2 imply that the holomorphic sections χ and ξ are continuous and belong to the Sobolev space $W^{1,2}_{\text{loc}}(\Omega, \mathbb{H})$. Hence for non-vanishing χ the potentials U and B belong to $L^2_{\text{loc}}(\Omega)$ and $\begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1}$ is continuous and belong to $W^{1,2}_{\text{loc}}(\Omega, \mathbb{H})$. In this case the statements concerning $\begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \partial+B & \bar{U} \\ -U & \bar{\partial}+\bar{B} \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$ and $\begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix} = \begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1}$ follow from the foregoing calculations.

2. For $\chi = (\mathbf{1} + \mathbf{l}_\Omega(0) \begin{pmatrix} 0 & -\bar{A} \\ A & 0 \end{pmatrix})^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$ with $(a, b) \in \mathbb{P}$. We shall extend the arguments of step 1 with a limiting argument to small potentials $A \in L^2(\Omega)$. In fact, for any small $A \in L^2(\Omega)$ we choose a sequence A_n of smooth potentials in $L^2(\Omega)$ with limit A . We extend all potentials to a slightly larger open domain Ω' which contains the closure $\bar{\Omega}$, so that they vanish on the relative complement of Ω in Ω' . Obviously, the corresponding sequence of spinors χ_n defined above extend to Ω' . By definition these spinors are smooth on $\Omega' \setminus \bar{\Omega}$. Furthermore, the sequence of integrals of the corresponding one-forms $B_n dz$ along a closed path in $\Omega' \setminus \bar{\Omega}$ around Ω converges. Since the sequence of measures $\frac{1}{2\sqrt{-1}} A_n \bar{A}_n d\bar{z} \wedge dz$ converges, this implies that the sequence U_n is a bounded sequence in $L^2(\Omega)$. Due to the Banach–Alaoglu theorem [R-S-I, Theorem IV.21], this sequence U_n has a weakly convergent subsequence with limit U . Also the sequence of real signed measures $\frac{1}{2\sqrt{-1}} (A_n \bar{A}_n - U_n \bar{U}_n) d\bar{z} \wedge dz$ on Ω has a weakly convergent subsequence. Finally, due to the equations $\bar{\partial} B = A \bar{A} - U \bar{U}$, the sequence of functions B_n is bounded in the *Lorentz space* $L^{2,\infty}(\Omega)$. Due to [B-S, Chapter 2 Theorem 2.7. and Chapter 4 Corollary 4.8.] this *Lorentz space* is the dual space of the corresponding *Lorentz space* $L^{2,1}(\Omega)$. The sequence B_n has also a weakly convergent subsequence with limit B and $\bar{\partial} B$ considered as a measure is equal to the limit of the measures $\frac{1}{2\sqrt{-1}} (A_n \bar{A}_n - U_n \bar{U}_n) d\bar{z} \wedge dz$. Since the sequence of spinors χ_n converges in $L^q(\Omega, \mathbb{H})$, and since the sequences U_n and B_n both converge weakly, the limit χ is anti-holomorphic with respect to the anti-holomorphic structure defined by the limits U and B .

Next we prove that the function $\bar{\partial} B = -\bar{\partial} \partial \ln(\chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2)$ considered as a measure contains no point measures.

Lemma 5.4. *If Ω denotes a bounded open subset of \mathbb{C} , then for all finite signed Baire measures $d\mu$ on Ω [Ro, Chapter 13 Section 5] there exists a function h in the Zygmund space*

$L_{\text{exp}}(\Omega)$ [B-S, Chapter 4 Section 6.] such that $-\bar{\partial}\partial h = d\mu$ (in the sense of distributions). Moreover, if for a suitable $\varepsilon > 0$ all ε -balls of Ω have measure smaller than π/q with respect to the positive part $d\mu^+$ of the Hahn decomposition of the finite signed Baire measure $d\mu$ on Ω [Ro, Chapter 11 Section 5], then the exponentials $\exp(h)$ of all $h \in L_{\text{exp}}(\Omega)$ with $-\bar{\partial}\partial h = d\mu$ belong to $L_{\text{loc}}^q(\Omega)$. Conversely, if the positive part $d\mu^+$ contains a point measure with mass π/q , then the corresponding functions $h \in L_{\text{exp}}(\Omega)$ with $-\bar{\partial}\partial h = d\mu$ do not belong to $L_{\text{loc}}^q(\Omega)$.

Proof. Due to Dolbeault's Lemma [Gu-Ro, Chapter I Section D 2. Lemma] the convolution with the function $-\frac{2}{\pi} \ln |z|$ defines a right inverse of the operator $-\bar{\partial}\partial$. Now we claim that the restriction of this convolution operator defines a bounded operator from $L^1(\Omega)$ into the Zygmund space $L_{\text{exp}}(\Omega)$. Since the domain Ω is bounded, the claim is equivalent to the analogous statement about the restriction to Ω of the convolution with the non-negative function

$$f(z) = \begin{cases} -\frac{2}{\pi} \ln |z| & \text{if } |z| < 1 \\ 0 & \text{if } 1 \leq |z| \end{cases}.$$

Associated to this function f is its distribution function μ_f and its non-increasing rearrangement f^* ([S-W, Chapter II §3. Chapter V §3.], [B-S, Chapter 2 Section 1.] and [Zi, Chapter 1. Section 8.]):

$$\mu_f(s) = \pi \exp(-\pi s) \qquad f^*(t) = \begin{cases} -\frac{\ln(t/\pi)}{\pi} & \text{if } 0 \leq t \leq \pi \\ 0 & \text{if } \pi \leq t \end{cases}.$$

If $g \in L^1(\Omega)$, then $g^{**}(t) = \frac{1}{t} \int_0^t g^*(s) ds$ is bounded by $\|g\|_1/t$, since the $L^{1,\infty}$ -norm $\|g\|_{(1,\infty)} = \sup\{tg^{**}(t) \mid t > 0\} = \int_0^\infty g^*(t) dt$ coincides with the L^1 -norm [S-W, Chapter V (3.9)]. Therefore, [Zi, (1.8.14) and (1.8.15)] in the proof of [Zi, 1.8.8. Lemma] (borrowed from [O, Lemma 1.5.]) implies that the non-increasing rearrangement $h^*(t)$ of the convolution $h = f * g$ is bounded by

$$\begin{aligned} h^*(t) \leq h^{**}(t) \leq h_2^{**}(t) + h_1^{**}(t) &\leq g^{**}(t) \int_{f^*(t)}^\infty \mu_f(s) ds - \int_t^\infty s g^{**}(s) df^*(s) \\ &\leq \frac{\|g\|_1}{t} \exp(-\pi f^*(t)) - \|g\|_1 \int_t^\infty df^*(s) \\ &\leq \frac{\|g\|_1}{\pi} + \|g\|_1 f^*(t). \end{aligned}$$

Since by definition the non-increasing rearrangement $h^*(t)$ vanishes for all arguments, which are larger than the Lebesgue measure of Ω , we conclude the validity of the following estimate:

$$\int_0^\infty (\exp(qh^*(t)) - 1) dt \leq \int_0^{|\Omega|} \exp(qh^*(t)) dt \leq |\Omega| \exp(q\|g\|_1/\pi) \int_0^{|\Omega|} (t/\pi)^{-\|g\|_1 q/\pi} dt,$$

with an obvious modification when $\pi < |\Omega|$. Due to a standard argument [B-S, Chapter 2 Exercise 3.] the finiteness of this integral is equivalent to the statement that $\exp|h|$ belongs to $L^q(\Omega)$. To sum up, the exponential $\exp(h)$ of the convolution $h = f * g$ belongs to $L^q(\Omega)$, if $q < \frac{\pi}{\|g\|_1}$. This proves the claim. In particular, for all $g \in L^1(\Omega)$ there exists an element $h \in L_{\exp}(\Omega)$ with $-\bar{\partial}\partial h = g$.

Due to [B-S, Chapter 4 Theorem 6.5] $L_{\exp}(\Omega)$ is the dual space of the Zygmund space $L \log L(\Omega)$. Hence we shall improve the previous estimate and show that the convolution with $-\frac{2}{\pi} \ln|z|$ defines a bounded operator from $L \log L(\Omega) \subset L^1(\Omega)$ into $C(\Omega) \subset L_{\exp}(\Omega)$. By definition of the norm [B-S, Chapter 4 Definition 6.3.]

$$\|g\|_{L \log L} = -\frac{1}{|\Omega|} \int_0^{|\Omega|} g^*(t) \ln(t/|\Omega|) dt = \int_0^{|\Omega|} g^{**}(t) dt$$

we may improve the previous estimate to [Zi, (1.8.14) and (1.8.15)]

$$h^{**}(t) \leq g^{**}(t) \int_{f^*(t)}^{\infty} \mu_f(s) ds - \int_t^{\infty} s g^{**}(s) df^*(s) \leq \frac{1}{\pi} \int_0^t g^*(s) ds + \int_t^{\pi} g^{**}(s) ds \leq \|g\|_{L \log L}.$$

This implies that in this case $h^{**}(t)$ is bounded, and consequently $h \in L^\infty(\Omega)$. Furthermore, since the function $\ln|z|$ is continuous for $z \neq 0$, the convolution with $-\frac{2}{\pi} \ln|z|$ is a bounded operator from $L \log L(\Omega)$ into the Banach space $C(\Omega)$. Finally, the dual of this operator yields a bounded operator from the Banach space of finite signed Baire measures on Ω [Ro, Chapter 13 Section 5 25. Riesz Representation Theorem] into $L_{\exp}(\Omega)$. More precisely, if the measure of Ω with respect to a finite positive measure $d\mu$ is smaller than π/q , then the exponential $\exp(h)$ of the corresponding function $h = f * d\mu$ belongs to $L^q(\Omega)$.

Due to Weyl's Lemma [R-S-II, Theorem IX.25] the difference of two arbitrary functions h_1 and h_2 with $-\bar{\partial}\partial h_1 = -\bar{\partial}\partial h_2 = d\mu$ is analytic. Therefore, it suffices to show the second and third statement of the lemma for the convolution of $-\frac{2}{\pi} \ln|z|$ with $d\mu$. Due to the boundedness of Ω we may neglect that part of this convolution, where the former function is negative. Therefore, we may neglect the negative part of $d\mu$ in order to bound the exponential $\exp(h)$. The decomposition of the convolution into an ε -near and an ε -distant part analogous to the decomposition in the proof of Lemma 3.2 completes the proof. **q.e.d.**

If the function $\bar{\partial}B$ considered as a finite Baire measure contains a point measure at $z = z'$ of negative mass smaller or equal to $-n\pi$, then, due to Lemma 5.4, the spinor $\tilde{\chi} = \begin{pmatrix} z-z' & 0 \\ 0 & \bar{z}-\bar{z}' \end{pmatrix}^{-n} \chi$ belongs to $\bigcap_{q < \infty} L_{\text{loc}}^q(\Omega, \mathbb{H})$. This implies that χ has a zero of order n at z' .

Hence, due to our assumptions, the masses of all point measures are larger than $-\pi$. Again the following Lemma 5.4 implies that $\begin{pmatrix} \chi_1 & \chi_2 \\ -\bar{\chi}_2 & \bar{\chi}_1 \end{pmatrix}^{-1}$ is a L_{loc}^2 -spinor in the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$. Since these kernels are contained in $\bigcap_{q < \infty} L_{\text{loc}}^q(\Omega, \mathbb{H})$, Lemma 5.4 implies that this measure contains no point measures.

Finally we show that $\begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \partial+B & \bar{U} \\ -U & \bar{\partial}+\bar{B} \end{pmatrix} \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$. For this purpose we use again the sequence of smooth potentials A_n in $L^2(\Omega)$ with limits A and the corresponding sequence of spinors χ_n on Ω' . We choose Ω small enough such that the corresponding sequences of potentials U_n belong to the subsets described in Theorem 3.2, on which the resolvents are weakly continuous. The arguments of Theorem 3.2 imply also that the sequence $\mathbf{l}_\Omega(U_n)$ considered as operators from $L^p(\Omega, \mathbb{H})$ into $L^q(\Omega, \mathbb{H})$ with $\frac{1}{p} < \frac{1}{q} + \frac{1}{2}$ converges to $\mathbf{l}_\Omega(U)$. Now for any quaternionic function f in $L^q(\Omega, \mathbb{H})$, the sequence of quaternionic functions $\begin{pmatrix} \partial+B_n & \bar{U}_n \\ -U_n & \bar{\partial}+\bar{B}_n \end{pmatrix} \mathbf{l}_\Omega(A_n)f$ belong on the complement of the support of f in Ω to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U}_n \\ U_n & \bar{\partial} \end{pmatrix}$. Therefore it satisfies on this complement the corresponding quaternionic version of *Cauchy's Integral Formula* 2.1. Due to continuity this implies that the limits obeys the quaternionic version of *Cauchy's Integral Formula* 2.1 in the sense of distributions on the complement of the support of f in Ω . Hence the limit belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$. Since the spinor ξ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \bar{\partial} \end{pmatrix}$ obey the corresponding quaternionic version of *Cauchy's Integral Formula* 2.1, we may represent it on any open subset, whose closure is contained in Ω , as $\xi = \mathbf{l}_\Omega(A)f$ with an appropriate f , whose support is disjoint from the open subset in Ω . Furthermore, ξ is the limit of $\mathbf{l}_\Omega(A_n)f$. This implies that ψ belongs on Ω to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$.

3. For general χ . Due to step 2, the quotient $\begin{pmatrix} \tilde{\chi}_1 & -\bar{\tilde{\chi}}_2 \\ \tilde{\chi}_2 & \bar{\tilde{\chi}}_1 \end{pmatrix}^{-1} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}$ of χ divided by the inverse of the $\tilde{\chi}$, which was considered in step 2 is continuous and belongs to $W_{\text{loc}}^{2,p}(\Omega)$. This implies that all components of the difference $\begin{pmatrix} \partial\chi_1 & -\partial\bar{\chi}_2 \\ \bar{\partial}\chi_2 & \bar{\partial}\bar{\chi}_1 \end{pmatrix} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 \\ \chi_2 & \bar{\chi}_1 \end{pmatrix}^{-1} - \begin{pmatrix} \partial\tilde{\chi}_1 & -\partial\bar{\tilde{\chi}}_2 \\ \bar{\partial}\tilde{\chi}_2 & \bar{\partial}\bar{\tilde{\chi}}_1 \end{pmatrix} \begin{pmatrix} \tilde{\chi}_1 & -\bar{\tilde{\chi}}_2 \\ \tilde{\chi}_2 & \bar{\tilde{\chi}}_1 \end{pmatrix}^{-1}$ belong to $\bigcap_{1 < p < 2} W_{\text{loc}}^{1,p}(\Omega) \times W_{\text{loc}}^{1,p}(\Omega)$. Now the arguments of step 2 carry over to all χ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \bar{\partial} \end{pmatrix}$ without zeroes on Ω .

4. Inverse transformation. The arguments of steps 1–3 carry over and show, that $\begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix}^{-1}$ belong on Ω to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \bar{\partial} \end{pmatrix}$. All other statements follow from direct calculations. **q.e.d.**

Actually we proved the following quaternionic version of

Weyl's Lemma 5.5. *Let $\begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix}$ be spinor without zeros in the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -U & \bar{\partial} \end{pmatrix}$ with potential $U \in L_{\text{loc}}^2(\Omega)$ on a domain $\Omega \subset \mathbb{C}$. Then a function $\begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} \in L_{\text{loc}}^p(\Omega, \mathbb{H})$ with $1 < p < 2$ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$ if $\begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix} \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$ is a closed current on Ω .*

Proof. Due to the assumptions there exists a function $f \in \bigcap_{r < p} W^{1,r}(\Omega, \mathbb{H})$ with

$$d \begin{pmatrix} f_1 & -\bar{f}_2 \\ f_2 & \bar{f}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix} \begin{pmatrix} dz & 0 \\ 0 & d\bar{z} \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}.$$

Now the *Bäcklund transformation* 5.3 implies that $\chi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix}^{-1}$ and $\xi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\bar{\phi}_2 & \bar{\phi}_1 \end{pmatrix}^{-1} f$ belong to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{A} \\ A & \bar{\partial} \end{pmatrix}$ with an appropriate $A \in L^2_{\text{loc}}(\Omega)$. Finally, again due to the *Bäcklund transformation* 5.3, ψ belongs to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$. **q.e.d.**

6 The Plücker formula

Let $H \subset H^0(X, \mathcal{Q}_D)$ be a quaternionic linear system in the space of holomorphic sections of a holomorphic quaternionic line bundle on a compact Riemann surface X . At any point $x \in X$ we have a sequence $\text{ord}_1 H < \dots < \text{ord}_{\dim H} H$ of *Orders of zeroes* 2.6 of elements of H , which differ only at finitely many points form the sequence $\text{ord}_1 H = 1, \dots, \text{ord}_{\dim H} H = \dim H$. The order of H is defined as [F-L-P-P, Definition 4.2.]:

$$\text{ord } H = \sum_{x \in X} \left(\text{ord}_1 H - 1 \right) + \dots + \left(\text{ord}_{\dim H} H - \dim H \right).$$

For smooth Hopf fields the following estimate is proven in [F-L-P-P, Corollary 4.8.]:

$$\frac{1}{2\pi\sqrt{-1}} \int_X Q \wedge \bar{Q} \geq \dim H ((1 - g)(\dim H - 1) - \deg(D)) + \text{ord } H.$$

We shall show that this inequality holds for all square integrable Hopf fields. For this purpose we fit together the local *Bäcklund transformation* 5.3 to a global transformation.

Corollary 6.1. *Let $\xi, \chi \in H^0(X, \mathcal{Q}_D)$ be two holomorphic spinors of a quaternionic holomorphic line bundle with Hopf field Q over the complex holomorphic line bundle corresponding to \mathcal{O}_D on a compact Riemann surface X . If χ has no zeroes, then the local Bäcklund transformation 5.3 induces a global Bäcklund Transformation $Q \mapsto \tilde{Q}$ from the Hopf field Q to an Hopf field \tilde{Q} of a quaternionic holomorphic line bundle over the complex holomorphic line bundle corresponding to \mathcal{O}_{D+K} and a paired quaternionic holomorphic line bundle over the complex holomorphic line bundle corresponding to \mathcal{O}_{-D} with two holomorphic sections, respectively. The Willmore functionals of these Hopf fields obey the equation*

$$\frac{1}{2\pi\sqrt{-1}} \int_X \left(Q \wedge \bar{Q} - \tilde{Q} \wedge \bar{\tilde{Q}} \right) = -\deg(D). \quad \mathbf{q.e.d.}$$

An $\dim(H)$ -fold application of this Corollary immediately implies the Plücker formula. Indeed, first we choose a member χ of the linear system H of lowest vanishing order at all points of X . Since the Riemann surface has complex dimension one, such sections of the quaternionic vector space H always exists. We change the divisor of the quaternionic holomorphic line bundle, such that χ is a section without zeroes of $H^0(X, \mathcal{Q}_D)$. An application of Corollary 6.1 with this χ transforms the linear system $H \subset H^0(X, \mathcal{Q}_D)$ into a linear system $\tilde{H} \subset H^0(X, \mathcal{Q}_{D+K})$ of (quaternionic) dimension $\dim H - 1$. We may repeat such an application of Corollary 6.1 until we end with a trivial linear system with Hopf field A . We remark

that the sum over the degrees of the corresponding sequence of quaternionic holomorphic line bundles is equal to $\deg(D) \dim H - \text{ord } H + \sum_{j=0}^{\dim H-1} j \deg(K)$. Consequently, these Hopf fields obey the formula

$$\begin{aligned} \frac{1}{2\pi\sqrt{-1}} \int_X (Q \wedge \bar{Q} - A \wedge \bar{A}) &= -\deg(D) \dim H + \text{ord } H - \sum_{j=0}^{\dim H-1} j \deg(K) \\ &= \dim H ((1-g)(\dim H - 1) - \deg(D)) + \text{ord } H. \end{aligned}$$

This implies the general

Plücker formula 6.2. *Let X be a compact Riemann surface and \mathcal{Q}_D the sheaf of holomorphic sections of a holomorphic structure with a square integrable Hopf field Q (i. e. $\frac{1}{2\sqrt{-1}} \int_X Q \wedge \bar{Q} < \infty$) on the quaternionic line bundle over the complex line bundle corresponding to \mathcal{O}_D . Then all linear systems $H \subset H^0(X, \mathcal{Q}_{K-D})$ obey*

$$\frac{1}{2\pi\sqrt{-1}} \int_X Q \wedge \bar{Q} \geq \dim H ((1-g)(\dim H - 1) - \deg(D)) + \text{ord } H. \quad \text{q.e.d.}$$

7 Weak limits of Hopf fields

In this section we consider sequences of non-trivial sections of sequences of holomorphic quaternionic line bundles over a compact Riemann surface X . If the degrees of the underlying complex line bundles and the Hopf fields are bounded, then these sequences have convergent subsequences.

Theorem 7.1. *Let ψ_n be a sequence of non-trivial holomorphic sections of a sequence of quaternionic line bundles over the holomorphic complex line bundles corresponding to \mathcal{O}_{D_n} with Hopf fields Q_n . If the sequence of degrees $\deg(D_n)$ is bounded, then the sequence of underlying holomorphic complex line bundles has a convergent subsequence. If in addition the sequence of Hopf fields is bounded (i. e. $\frac{1}{2\sqrt{-1}} \int_X Q_n \wedge \bar{Q}_n \leq C < \infty$), then the appropriate renormalized sequence ψ_n has a subsequence, which converges to a non-trivial holomorphic section of a holomorphic quaternionic line bundle over a holomorphic complex line bundle corresponding to \mathcal{O}_D , where $D - D_n$ converges to an effective divisor D' . Moreover, the Hopf fields is a weak limit of the Hopf fields of the holomorphic structures corresponding to $\mathcal{Q}_{D_n+D'}$.*

Proof. The proof precedes in five steps.

1. The decomposition of the sequence of Hopf fields. Due to the Banach–Alaoglu theorem [R-S-I, Theorem IV.21] and the Riesz Representation theorem [Ro, Chapter 13 Section 5] the sequence of bounded finite Baire measures $\frac{1}{2\sqrt{-1}} Q_n \wedge \bar{Q}_n$ on X has a convergent subsequence. The limit can have only finitely many points $\{x_1, \dots, x_L\}$, whose mass is larger

than the constant of Theorem 3.2. We shall decompose the sequence of Hopf fields Q_n into a sum

$$Q_n = Q_{\text{reg},n} + \sum_{l=1}^L Q_{\text{sing},n,l}$$

of Hopf fields with disjoint support. Here $Q_{\text{sing},n,1}, \dots, Q_{\text{sing},n,L}$ are the restrictions of Q_n to small disjoint balls $B(x_1, \varepsilon_{n,1}), \dots, B(x_L, \varepsilon_{n,L})$, whose radii $\varepsilon_{n,l}$ tend to zero. Consequently, $Q_{\text{reg},n}$ are the restrictions of Q_n to the complements of the union of these balls. More precisely, we assume

Decomposition (i) For all $l = 1, \dots, L$ the weak limit of the sequence of finite Baire measures $\frac{1}{2\sqrt{-1}}Q_{\text{sing},n,l} \wedge \bar{Q}_{\text{sing},n,l}$ [Ro, Chapter 13] is equal to the point measures of the weak limit of $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$ at x_l . Consequently, the weak limit of the measures $\frac{1}{2\sqrt{-1}}Q_{\text{reg},n} \wedge \bar{Q}_{\text{reg},n}$ is equal to the weak limit of the measures $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$ minus the corresponding point measures at x_1, \dots, x_L .

Obviously there are many possible choices of the sequences $\varepsilon_{n,l}$ with this property (e. g. for a unique choice of $\varepsilon_{n,l}$ the square of the L^2 -norm of $Q_{\text{sing},n,l}$ is equal to the mass of the point measure at x_l of the weak limit of $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$). Locally we may consider the Hopf fields $Q_{\text{sing},n,l}$ as Hopf fields on \mathbb{P} . We want to transform each of these L sequences of Hopf fields by Möbius transformations on \mathbb{P} , such that the transformed Hopf fields belong to a set of the form described in Theorem 3.2. The pullbacks under the inverse of the action of the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$ on \mathbb{P}

$$SL(2, \mathbb{C}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{P} \rightarrow \mathbb{P}, z \mapsto \frac{az + b}{cz + d}$$

yields a unitary representation of the Möbius group on the Hilbert space of square integrable Hopf fields. In doing so we transform each of these L sequences of Hopf fields $Q_{\text{sing},n,l}$ (considered as Hopf fields on \mathbb{P}) by a sequence of Möbius transformations $g_{n,l}$ in such a way that the transformed sequence of Hopf fields has the following property:

Decomposition (ii) There exists some $\varepsilon > 0$, such that the L^2 -norm of the restrictions of the transformed Hopf fields $(g_{n,l}^{-1})^* Q_{\text{sing},n,l}$ to all ε -balls (with respect to the usual metric of \mathbb{P}) is bounded by the constant $C_p < S_p^{-1}$.

Such decompositions do not always exist. But we shall see now that, if all points of \mathbb{P} have measure smaller than $2S_p^{-2}$ with respect to the weak limit of the finite Baire measures $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$, then these decompositions indeed exist. The free Dirac operator on \mathbb{P} is invariant under the compact subgroup $SU(2, \mathbb{C})/\mathbb{Z}_2$ of the Möbius group ($\simeq SL(2, \mathbb{C})/\mathbb{Z}_2$) as well as the usual metric on \mathbb{P} . Therefore, due to the global Iwasawa decomposition [He, Chapter VI Theorem 5.1], it suffices to consider in place of the whole Möbius group only the semidirect product of the scaling transformations ($z \mapsto \exp(t)z$ with $t \in \mathbb{R}$) with the translations ($z \mapsto z + z_0$ with $z_0 \in \mathbb{C}$). In the sequel we assume that all $g_{n,l}$ belong to this subgroup of the Möbius group.

Lemma 7.2. *If the square of the L^2 -norm of a Hopf field Q on \mathbb{P} is smaller than $2S_p^{-2}$, then there exists a constant $C_p < S_p^{-1}$ and a Möbius transformation such that the L^2 -norm of the restrictions of the transformed Hopf fields to all balls of radius $\pi/6$ is not larger than C_p .*

Proof. Let $r_{\max}(Q)$ be the maximum of the set

$$\{r \mid \text{the } L^2\text{-norms of the restrictions of } Q \text{ to all balls of radius } r \text{ are not larger than } C_p\}.$$

Since the L^2 -norm of the restriction of Q to a ball depends continuously on the center and the diameter of the ball, this set has indeed a maximum. Moreover, there exist balls with radius $r_{\max}(Q)$, on which the restricted Hopf field has L^2 -norm equal to C_p .

We claim that there exists a Möbius transformation h , such that $r_{\max}((h^{-1})^*Q)$ is the supremum of the set of all $r_{\max}((g^{-1})^*Q)$, where g runs through the Möbius group. Let g_n be a maximizing sequence of this set, i. e. the limit of the sequence $r_{\max}((g_n^{-1})^*Q)$ is equal to the supremum of the former set. Since $r_{\max}((g^{-1})^*Q)$ is equal to $r_{\max}(Q)$, if g belongs to the subgroup $SU(2, \mathbb{C})/\mathbb{Z}_2$ of isometries of the Möbius group, and due to the global Iwasawa decomposition [He, Chapter VI Theorem 5.1], the sequence g_n may be chosen in the semidirect product of the scaling transformations $z \mapsto \exp(t)z$ with the translations $z \mapsto z + z_0$. It is quite easy to see, that if the values of t and z_0 corresponding to a sequence g_n of such Möbius transformations are not bounded, then there exist arbitrary small balls, on which the L^2 -norms of the restrictions of $(g_n^{-1})^*Q$ have subsequences converging to the L^2 -norm of Q . Hence if the L^2 -norm of Q is larger than C_p , then the maximizing sequence of Möbius transformations can be chosen to be bounded and therefore has a convergent subsequence. In this case the continuity implies the claim. If the L^2 -norm of Q is not larger than C_p , then $r_{\max}((g^{-1})^*Q)$ does not depend on g and the claim is obvious.

If the L^2 -norm of Q is smaller than $\sqrt{2}C_p$, then all $r_{\max}(Q)$ -balls, on which the restriction of Q has L^2 -norm equal to C_p , have pairwise non-empty intersection. In particular, all of them have non-empty intersection with one of these balls. Consequently, if $r_{\max}(Q)$ is smaller than $\pi/6$, then these $r_{\max}(Q)$ -balls are contained in one hemisphere. In this case there exists a Möbius transformation g , which enlarges $r_{\max}(Q)$ (i. e. $r_{\max}(Q) < r_{\max}((g^{-1})^*Q)$). We conclude that there exist a Möbius transformation g , such that $r_{\max}((g^{-1})^*Q)$ is smaller than $\pi/6$. **q.e.d.**

The upper bound $2S_p^{-2}$ is sharp, because for a sequence of L^2 -Hopf fields on \mathbb{P} , whose square of the absolute values considered as a sequence of finite Baire measures converges weakly to the sum of two point measures of mass S_p^{-2} at opposite points, the corresponding sequence of maxima of $r_{\max}((h^{-1})^* \cdot)$ converges to zero. But the lower bound $\pi/6$ is of course not optimal.

If the L^2 -norms of the Hopf fields $Q_{\text{sing},n,l}$ are smaller than $\sqrt{2}C_p$, then this lemma ensures the existence of Möbius transformations $g_{n,l}$ with the property *Decomposition (ii)*. In general we showed the existence of a sequence of Möbius transformations $h_{n,l}$, which maximizes $r_{\max}((g^{-1})^*Q_{\text{sing},n,l})$. If for one $l = 1, \dots, L$ the corresponding sequences $(h_{n,l}^{-1})^*Q_{\text{sing},n,l}$ do not obey condition *Decomposition (ii)*, then we apply this procedure of decomposition to the corresponding sequence of Hopf fields $(h_{n,l}^{-1})^*Q_{\text{sing},n,l}$ on \mathbb{P} . Consequently, we decompose the

sequence $Q_{\text{sing},n,l}$ into a finite sum of Hopf fields with disjoint support, such that the corresponding Hopf fields $(g_{n,l}^{-1})^* Q_{\text{sing},n,l}$ on \mathbb{P} obey the analogous conditions *Decomposition* (i). Due to Lemma 7.2 after finitely many iterations of this procedure of decomposing the Hopf fields into finite sums of Hopf fields with disjoint support, we arrive at a decomposition

$$Q_n = Q_{\text{reg},n} + \sum_{l=1}^{L'} Q_{\text{sing},n,l}$$

of Hopf fields with disjoint support. More precisely, the Hopf fields $Q_{\text{sing},n,1}, \dots, Q_{\text{sing},n,L'}$ are restrictions of Q_n either to small balls or to the relative complements of finitely many small balls inside of small balls. In particular, the domains of these Hopf fields are excluded either from the domain of $Q_{\text{reg},n}$, or from the domain of another $Q_{\text{sing},n,l}$. The former Hopf fields obey condition *Decomposition* (i) and the latter obey condition

Decomposition (i') If the domain of $Q_{\text{sing},n,l'}$ is excluded from the domain of $Q_{\text{sing},n,l}$, then the weak limit of the sequence of finite Baire measures $\frac{1}{2\sqrt{-1}} (g_{n,l}^{-1})^* Q_{\text{sing},n,l'} \wedge \bar{Q}_{\text{sing},n,l}$ on $\mathbb{C} \subset \mathbb{P}$ converges weakly to the point measure of the weak limit of the sequence $\frac{1}{2\sqrt{-1}} (g_{n,l}^{-1})^* Q_{\text{sing},n,l} \wedge \bar{Q}_{\text{sing},n,l}$ at some point of \mathbb{C} , whose measure with respect to the latter limit is not smaller than S_p^{-2} .

All these Hopf fields $Q_{\text{sing},n,1}, \dots, Q_{\text{sing},n,L'}$ obey condition *Decomposition* (ii). We remark that if the weak limit of the finite Baire measures $\frac{1}{2\sqrt{-1}} (h_{n,l}^{-1})^* Q_{\text{sing},n,l} \wedge \bar{Q}_{\text{sing},n,l}$ on $\mathbb{C} \subset \mathbb{P}$, where $h_{n,l}$ denotes the sequence of Möbius transformations maximizing $r_{\max}((g^{-1})^* Q_{\text{sing},n,l})$, contains at $z = \infty$ a point measure, whose mass is not smaller than S_p^{-2} , then we decompose from the sequence $(h_{n,l}^{-1})^* Q_{\text{sing},n,l}$ Hopf fields, whose domains are the complement of a large ball in the domains of these Hopf fields. In these cases the domains of the analog to the regular sequence of the decomposition are excluded from the domains of the analog to the singular sequence, whose L^2 -norm accumulates at $z = \infty$. Since the Möbius transformations corresponding to the former are faster divergent than the Möbius transformations of the latter, the latter should be considered as less singular than the former. Therefore, also in this case the domains of the more singular sequences are excluded from the domains of the less singular sequences. To sum up, the sequence $Q_{\text{reg},n}$ of Hopf fields on X and the sequences $(g_{n,1}^{-1})^* Q_{\text{sing},n,1}, \dots, (g_{n,L'}^{-1})^* Q_{\text{sing},n,L'}$ of Hopf fields on \mathbb{P} belong to a set of the form described in Theorem 3.2.

2. Limits of the sequence of underlying holomorphic complex line bundles. Due to the Banach–Alaoglu theorem [R-S-I, Theorem IV.21] and the Riesz Representation theorem [Ro, Chapter 13 Section 5] the sequence of finite Baire measures $\frac{1}{2\sqrt{-1}} Q_n \wedge \bar{Q}_n$ on X has a convergent subsequence. By passing to a subsequence we achieve that the sequence of finite Baire measures $\frac{1}{2\sqrt{-1}} Q_n \wedge \bar{Q}_n$ weakly converges. Since every divisor D of bounded degree is equivalent to the difference $D \sim D' - D''$ of two effective divisors D' and D'' of bounded degrees (compare [Fo, Theorem 21.7.]) a subsequence of the sequence of divisors D_n is equivalent to a convergent sequence of divisors with limit D . By passing to an equivalent subsequence we achieve that the sequence of divisors D_n converges to the divisor D .

We cover X by open subsets

$$X = \mathbf{U}_0 \cup \dots \cup \mathbf{U}_L.$$

Here \mathbf{U}_0 is the complement of the union of small neighbourhoods of the support of the divisor D with the support of the divisor D_{spin} of the spin bundle used in Theorem 3.2 and all those points, whose mass with respect to the weak limit of the measure $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$ is greater or equal than the constant S_p^{-1} . The other sets $\mathbf{U}_1, \dots, \mathbf{U}_L$ are small open disjoint disks, which cover the connected components of the complement of \mathbf{U}_0 . Since the holomorphic structures of $\mathcal{Q}_{D_{\text{spin}}}$ are Dirac operators with potentials, whose resolvents are investigated in Theorem 3.2, the restrictions of the holomorphic structures to \mathbf{U}_0 is also of this form. Due to Theorem 3.2 and Lemma 3.3 the resolvents of these restrictions of the holomorphic structures to \mathbf{U}_0 converges. By subtracting from \mathbf{U}_0 additional small closed disks contained in additional open sets $\mathbf{U}_{L+1}, \dots, \mathbf{U}_{L'}$, which are disjoint from $\mathbf{U}_1, \dots, \mathbf{U}_L$ and form each other, we may achieve that the corresponding limit of the sequence of restrictions of the holomorphic structures to \mathbf{U}_0 has a resolvent. Due to Theorem 3.2 these restrictions of holomorphic structures have always reduced resolvents on the complement of a finite-dimensional subspace of holomorphic sections. Our arguments in step 5, where we prove the existence of convergent subsequences can be extended to this more general situation, since all bounded subsets of these finite-dimensional subsets are compact.

3. Limits of the local resolvents near the singular points with trivial kernels of the blown up holomorphic structures. In this step we consider the limits of the restrictions of the holomorphic structures to $\mathbf{U}_1, \dots, \mathbf{U}_L$. We assume that local parameters maps these small open disks onto small open domains in \mathbb{C} . Therefore the restrictions of the holomorphic structures to $\mathbf{U}_1, \dots, \mathbf{U}_L$ can be described by Dirac operators with potentials $\begin{pmatrix} U & \partial \\ -\bar{\partial} & \bar{U} \end{pmatrix}$ on open sets of \mathbb{C} . If \mathbf{U}_l does not contain a point, whose mass with respect to the weak limit of the measures $\frac{1}{2\sqrt{-1}}Q_n \wedge \bar{Q}_n$ is greater or equal than S_p^{-1} , then due to Theorem 3.2 and Lemma 3.3 the resolvents of the restrictions of the holomorphic structures to \mathbf{U}_0 converges to the resolvent of the holomorphic structure, whose Hopf field is the weak limit.

Let us assume that the support of the sequence of Hopf fields $Q_{\text{sing},n,l} = -\bar{U}_{\text{sing},n,l}d\bar{z}$ is contained in \mathbf{U}_l , and that the sequence of holomorphic structures with Hopf fields $(g_{n,1}^{-1})^* Q_{\text{sing},n,1}$ on \mathbb{P} has a trivial kernel. We claim, that in this case the corresponding sequence of resolvents of Dirac operators on \mathbf{U}_l , whose Hopf fields are given by the restrictions of the Hopf fields

$$Q_{\text{reg},n} + Q_{\text{sing},n,l} = -\bar{U}_{\text{reg},n}d\bar{z} - \bar{U}_{\text{sing},n,l}d\bar{z}$$

to \mathbf{U}_l , considered as operators from $L^p(\mathbf{U}_l) \times L^p(\mathbf{U}_l)$ into $L^q(\mathbf{U}_l) \times L^q(\mathbf{U}_l)$ with $1 < p < 2$ and $1 < q < \frac{2p}{2-p}$ converges to the resolvent of the Dirac operator, whose potential corresponds to

the weak limit of the sequence of Hopf fields. The corresponding resolvents obey the relation

$$\begin{aligned}
R_{\mathbb{C}}(U_{\text{reg},n} + U_{\text{sing},n,l}, \bar{U}_{\text{reg},n} + \bar{U}_{\text{sing},n,l}, 0) &= \\
&= R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \left(\mathbf{1} - \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \right)^{-1} \\
&= R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) + R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \\
&\quad \left(\mathbf{1} - \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \right)^{-1} \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0).
\end{aligned}$$

The operators

$$\begin{aligned}
&\left(\mathbf{1} - \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \right)^{-1} \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} \\
&= \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} \left(\mathbf{1} - R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} \right)^{-1}
\end{aligned}$$

depend only on the restrictions of $R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0)$ to the support of $U_{\text{sing},n,l}$. We shall transform this sequence of operators under the corresponding sequence of Möbius transformations $g_{n,l}$. The small open sets $\mathbf{U}_1, \dots, \mathbf{U}_L$ are identified with bounded open sets of \mathbb{C} . Therefore the restrictions of the holomorphic structures may be described by Dirac operators with potentials on bounded open sets of \mathbb{C} . All Möbius transformations h induce isometries

$$\mathbf{l}_p(h) : \quad L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C}) \quad f \mapsto \tilde{f} \quad \tilde{f}(z) = f(h^{-1}z) \left| \frac{dh^{-1}z}{dz} \right|^{\frac{2}{p}}.$$

A direct calculation shows that the resolvent $R_{\mathbb{C}}(0, 0, 0)$ of the free Dirac operator, considered as an operator from $L^p(\mathbb{C}) \times L^p(\mathbb{C})$ into $L^{\frac{2p}{2-p}}(\mathbb{C}) \times L^{\frac{2p}{2-p}}(\mathbb{C})$ with $1 < p < 2$ is invariant under the scaling transformations ($z \mapsto \exp(t)z$ with $t \in \mathbb{R}$) and the translations ($z \mapsto z + z_0$ with $z_0 \in \mathbb{C}$). Since these sequences of Möbius transformations belong to the semidirect product of the scaling transformations with the translations, the free resolvent is invariant under these transformations $g_{n,l}$:

$$\begin{pmatrix} \mathbf{l}_{\frac{2p}{2-p}}(g_{n,l}) & 0 \\ 0 & \mathbf{l}_{\frac{2p}{2-p}}(g_{n,l}) \end{pmatrix} R_{\mathbb{C}}(0, 0, 0) \begin{pmatrix} \mathbf{l}_p(g_{n,l}^{-1}) & 0 \\ 0 & \mathbf{l}_p(g_{n,l}^{-1}) \end{pmatrix} = R_{\mathbb{C}}(0, 0, 0).$$

If the sets $\mathbf{U}_1, \dots, \mathbf{U}_L$ are small, then the restrictions of the sequence of transformed Hopf fields $(g_{n,l}^{-1})^* Q_n$ to the subset $g_{n,l}^{-1}(\mathbf{U}_l) \subset \mathbb{P}$ still obey the condition of Lemma 3.3. Therefore, due to Theorem 3.2, the corresponding sequence of resolvents on \mathbb{P} converges. We assume that the limit is the resolvents of a Dirac operator on \mathbb{P} without kernel. In this case the arguments of Theorem 3.2 together with the first resolvent formula [R-S-I, Theorem VI.5]:

$$(\lambda - \lambda') R_{\lambda'} = R_{\lambda} \left(\frac{\mathbf{1}}{\lambda - \lambda'} - R_{\lambda} \right)^{-1} = \left(\frac{\mathbf{1}}{\lambda - \lambda'} - R_{\lambda} \right)^{-1} R_{\lambda},$$

imply that the corresponding sequence of resolvents considered as operators from $L^p(\mathbb{C}) \times L^p(\mathbb{C})$ into $L^{\frac{2p}{2-p}}(\mathbb{C}) \times L^{\frac{2p}{2-p}}(\mathbb{C})$ is bounded. We conclude that the sequences of operators

$$\begin{aligned} & \left(\mathbf{1} - \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \right)^{-1} \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} \\ &= \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} \left(\mathbf{1} - R_{\mathbb{C}}(U_{\text{reg},n}, \bar{U}_{\text{reg},n}, 0) \begin{pmatrix} U_{\text{sing},n,l} & 0 \\ 0 & \bar{U}_{\text{sing},n,l} \end{pmatrix} \right)^{-1} \end{aligned}$$

are bounded.

Due to Hölder's inequality [R-S-I, Theorem III.1 (c)] for $1 \leq q' < q \leq \infty$ the restriction of $L^q(\mathbb{C})$ into $L^{q'}(B(0, \varepsilon))$ is bounded by $(\pi \varepsilon^2)^{\frac{1}{q'} - \frac{1}{q}}$. Since the radii of the supports of $U_{\text{sing},n,l}$ tend to zero, the restrictions of the resolvents $R_{\mathbb{C}}(U_n, \bar{U}_n, 0)$ considered as operators from $L^p(\mathbb{U}_l) \times L^p(\mathbb{U}_l)$ into $L^q(\mathbb{U}_l) \times L^q(\mathbb{U}_l)$ with $1 < p < 2$ and $1 < q < \frac{2p}{2-p}$ converge to the resolvent of the weak limit of the sequences U_n .

4. Limits of the local resolvents near the singular points with non-trivial kernels of the blown up holomorphic structures. In this case we add to the sequence of divisors D_n a sequence of effective divisors D'_n with support in the complements of \mathbb{U}_0 , such that the corresponding transformed sequences of holomorphic structures corresponding with Hopf fields $(g_{n,1}^{-1})^* Q_{\text{sing},n,1}, \dots, (g_{n,L'}^{-1})^* Q_{\text{sing},n,L'}$ on \mathbb{P} have trivial kernels.

Lemma 7.3. *For any holomorphic quaternionic line bundle on \mathbb{P} with non-trivial kernel let d be the unique natural number such that*

$$\dim H^0(\mathbb{P}, \mathcal{Q}_{D-d\infty}) = 0 \qquad \dim H^0(\mathbb{P}, \mathcal{Q}_{D-(d-1)\infty}) = 1$$

Then there exists an effective divisor D' of degree $d - \deg(D) - 1$, whose support is contained in \mathbb{C} , such that

$$\dim H^0(\mathbb{P}, \mathcal{Q}_{D+D'+(n-d)\infty}) = n \qquad \forall n \in \mathbb{N}_0.$$

Proof. Due to the *Riemann–Roch Theorem* 4.2 we have the inequality

$$\dim H^0(\mathbb{P}, \mathcal{Q}_{D-d\infty}) = \deg(D) + 1 - d + \dim H^1(\mathbb{P}, \mathcal{Q}_{D-d\infty}) \geq \deg(D) + 1 - d.$$

By definition of d this implies $\deg(D) \leq d - 1$. Moreover, the equality $\deg(D) = d - 1$ is equivalent to $\dim H^1(\mathbb{P}, \mathcal{Q}_{D-d\infty}) = 0$. Due to *Sérre Duality* 4.1 this is equivalent to

$$\dim H^1(\mathbb{P}, \mathcal{Q}_{D+(n-d)\infty}) = 0 \qquad \forall n \in \mathbb{N}_0.$$

Finally, due to the *Riemann–Roch Theorem* 4.2, the equality $\deg(D) = d - 1$ is equivalent to

$$\dim H^0(\mathbb{P}, \mathcal{Q}_{D+(n-d)\infty}) = n \qquad \forall n \in \mathbb{N}_0.$$

Therefore it suffices to consider the cases $\deg(D) < d - 1$.

We claim that in this case there exists an element $z \in \mathbb{C}$, such that the analogous number d corresponding to the holomorphic structure of \mathcal{Q}_{D+z} is equal to d . This is equivalent

to $\dim H^0(\mathbb{P}, \mathcal{Q}_{D+z-d\infty}) = 0$. Let us assume on the contrary that for all $z \in \mathbb{C}$ we have $\dim H^0(\mathbb{P}, \mathcal{Q}_{D+z-d\infty}) = 1$. Consequently, for all pairwise different $z_1, \dots, z_L \in \mathbb{C}$ the dimension of the linear system $H^0(\mathbb{P}, \mathcal{Q}_{D+z_1+\dots+z_L-d\infty})$ is larger than L . For large L due to *Sérre Duality* 4.1, *Plücker formula* 6.2 the Čech cohomology group $H^1(\mathbb{P}, \mathcal{Q}_{D+z_1+\dots+z_L-d\infty})$ is trivial. Consequently, due to *Riemann–Roch Theorem* 4.2, we obtain

$$L \leq H^0(\mathbb{P}, \mathcal{Q}_{D+z_1+\dots+z_L-d\infty}) = 1 + \deg(D) + L - d,$$

which contradicts to $\deg(D) < d - 1$. This proves the claim.

By an iterated application of this claim we obtain an effective divisor D' with the desired properties. **q.e.d.**

We apply this lemma to the holomorphic structure corresponding to the weak limits of Hopf fields $(g_{n,1}^{-1})^* Q_{\text{sing},n,1}, \dots, (g_{n,L'}^{-1})^* Q_{\text{sing},n,L'}$ on \mathbb{P} . Since we are only interested in the restrictions of the holomorphic structure to $\mathbb{U}_1, \dots, \mathbb{U}_L$, we may change the degree at ∞ . For all holomorphic quaternionic line bundles on \mathbb{P} , with sheaf \mathcal{Q}_D of holomorphic sections, the sheaf of holomorphic sections $\mathcal{Q}_{D+D'-d\infty}$ of the corresponding holomorphic structure on the spin bundle has a trivial kernel. Here D' denotes the divisor of degree $d - \deg(D) - 1$ constructed in Lemma 7.3. Obviously, the sequence of divisors $D'_n = g_{n,l}(D')$ converge to the divisor $\deg(D')x_l$ on \mathbb{U}_l . Hence the arguments of step 4 imply that the resolvents of the corresponding Dirac operators on \mathbb{U}_l converges to the resolvent of the Dirac operator, whose potential is the weak limit of the sequence of potentials.

5. Limits of the sequence of holomorphic sections. In the preceding step we added to the sequence of divisors a sequence of convergent effective divisors. Obviously, any sequence of sections of the original sequence of holomorphic quaternionic line bundles are also holomorphic sections of the latter sequence of holomorphic quaternionic line bundles. We shall prove that this sequence converges to a non-trivial section of the limit of the latter sequence of holomorphic quaternionic line bundles. More precisely, the Hopf field of the limit of the holomorphic structures is the weak limit of the sequence of Hopf fields of the latter sequence of Hopf fields.

At the end of step 2 we saw that the sequence of resolvents of the restrictions of the holomorphic structures to \mathbb{U}_0 converged as an operator from $H^0(\mathbb{U}_0, \mathcal{L}_{D_{\text{spin}}}^p)$ into $H^0(\mathbb{U}_0, \mathcal{L}_{D_{\text{spin}}}^q)$ with $1 < p < 2$ and $1 < q < \frac{2p}{2-p}$. Moreover, in steps 3–5 we showed that for all $l = 1, \dots, L$ the resolvents of the restriction of the holomorphic structures to \mathbb{U}_l converged as an operator from $H^0(\mathbb{U}_0, \mathcal{L}_{D_{\text{spin}}}^p)$ into $H^0(\mathbb{U}_0, \mathcal{L}_{D_{\text{spin}}}^q)$ with $1 < p < 2$ and $1 < q < \frac{2p}{2-p}$.

Due to quaternionic version of *Cauchy's Integral Formula* 2.1 the holomorphic sections are uniquely determined by their restrictions to

$$(\mathbb{U}_1 \cap \mathbb{U}_0) \cup \dots \cup (\mathbb{U}_L \cap \mathbb{U}_0).$$

Moreover, for all $1 < p < 2$ and $1 < q < \frac{2p}{2-p}$ the $H^0(X, \mathcal{L}_{D_n}^q)$ -norms are uniformly bounded in terms of the $H^0(X, \mathcal{L}_{D_n}^p)$ -norms of the restrictions to $(\mathbb{U}_1 \cap \mathbb{U}_0) \cup \dots \cup (\mathbb{U}_L \cap \mathbb{U}_0)$. Since the sequence of Hopf fields is bounded, the $H^0(X, \mathcal{W}_{D_n}^{1,p})$ -norms are bounded uniformly in terms

of the $H^0\left(X, \mathcal{L}_{D_n}^{\frac{2p}{2-p}}\right)$ -norms. Now Kondrakov's Theorem [Au, Theorem 2.34] implies that any sequence of non-trivial eigenfunctions, whose $H^0(X, \mathcal{L}_{D_n}^q)$ -norms are equal to one, have a convergent subsequence, and that the limit is non-trivial. Due to the convergence of the resolvents in steps 2–4 the limit is holomorphic with respect to the holomorphic structures, whose Hopf field is the weak limit of the sequence of Hopf fields. **q.e.d.**

8 Existence of minimizers

In this section we prove the existence of minimizing surfaces in \mathbb{R}^3 and \mathbb{R}^4 of the Willmore functional inside all conformal classes. More precisely, we show that any sequence of conformal mappings from a compact Riemann surface X into \mathbb{R}^3 (or \mathbb{R}^4), whose Willmore functionals is bounded, may be transformed by a sequence of conformal mappings of $\mathbb{R}^3 \subset S^3$ (or $\mathbb{R}^4 \subset S^4$) into a sequence, which converges with respect to the $W^{2,p}(X)$ -topology for all $1 < p < 2$. Essentially this follows from the *Quaternionic Weierstraß Representation* 1.1 and Theorem 7.1. In [P-P] the global Weierstraß representation was generalized to conformal mappings into \mathbb{R}^4 . In fact, the ‘quaternionic function theory’ provides two version of a global Weierstraß representation into \mathbb{R}^4 . From our point of view they are related by a *Bäcklund transformation* 5.3.

Proposition 8.1. *For any sequence of mappings in one of the following classes there exists a sequence of conformal transformations of the target space, such that the transformed sequence has a convergent subsequence with respect to the topologies of $\bigcap_{1 < p < 2} W^{2,p}(X)$:*

- (i) *Smooth conformal mappings from a compact Riemann surface X into \mathbb{R}^3 with bounded Willmore functional.*
- (ii) *Smooth conformal mappings from a compact Riemann surface X into \mathbb{R}^4 with bounded Willmore functional.*
- (iii) *Smooth conformal mappings from a compact Riemann surface X into \mathbb{R}^4 with a fixed complex holomorphic line bundle underlying the quaternionic holomorphic line bundle (compare [P-P, Theorem 4.3]) and bounded Willmore functional.*

Proof. We use the *Quaternionic Weierstraß Representation* 1.1 [P-P, B-F-L-P-P] and its reduction to conformal mappings into the pure imaginary quaternions $\simeq \mathbb{R}^3$ [Ta-1, Ta-2, Fr-2]. Hence all immersion are represented by two non-trivial spinors of two paired holomorphic quaternionic line bundles. Let ϕ_n and ψ_n be the sequences of paired spinors corresponding to a minimizing sequence of the Willmore functional on the space of conformal immersions of a compact Riemann surface X into \mathbb{R}^3 or \mathbb{R}^4 . The *Plücker formula* 6.2 implies that the degrees of the corresponding quaternionic holomorphic line bundles are bounded from below. Since these two line bundles are paired, the degrees are also bounded from above. Therefore Theorem 7.1 implies that both sequences have convergent subsequences. But it might happen that the corresponding limits of the holomorphic structures have singularities. In this case

the degrees of the limits are not the limits of the degrees. The corresponding Weierstraß representations describe immersions of X into $S^3 \supset \mathbb{R}^3$ or $S^4 \supset \mathbb{R}^4$. But a conformal transformation of $S^3 \supset \mathbb{R}^3$ or $S^4 \supset \mathbb{R}^4$ transforms these immersions into immersions into \mathbb{R}^3 or \mathbb{R}^4 . We remark that the conformal transformations of $S^4 \supset \mathbb{R}^4$ are very easy to describe with the help of the *Bäcklund transformation* 5.3. In fact the conformal transformations are just equal to the action of $GL(2, \mathbb{Q})$ on the corresponding quaternionic two-dimensional subspace of holomorphic sections of the Bäcklund transformed quaternionic holomorphic line bundle. Observe that in Theorem 7.1 we implicitly use translations and rotations of the immersions corresponding to rescalings of the two holomorphic spinors of the two paired quaternionic holomorphic line bundles. If we use in addition some inversions, we may always achieve that the limit stays inside of \mathbb{R}^3 or \mathbb{R}^4 . The corresponding two limits of ϕ_n and ψ_n does not have poles. Consequently the quaternionic holomorphic line bundles have degrees equal to the limits of the corresponding sequences of degrees and they are paired. In case the underlying holomorphic complex line bundles are fixed, the limits of the quaternionic holomorphic line bundles have also these underlying complex holomorphic line bundles. **q.e.d.**

We consider this Proposition as Montel's Theorem of 'quaternionic function theory'. It implies the existence of minimizers of the Willmore functional.

Theorem 8.2. *The Willmore functional attains a minimum on the following classes:*

- (i) *Smooth conformal mappings from a compact Riemann surface into \mathbb{R}^3 .*
- (ii) *Smooth conformal mappings from a compact Riemann surface into \mathbb{R}^4 .*
- (iii) *Smooth conformal mappings from a compact Riemann surface into \mathbb{R}^4 with a fixed complex holomorphic line bundle underlying the quaternionic holomorphic line bundle (compare [P-P, Theorem 4.3]).*

Proof. Proposition 8.1 implies the convergence of a minimizing sequence in the enlarged classes (i)–(iii) of not necessarily smooth conformal mappings with bounded Willmore functional. It remains to ensure the smoothness of the minimizers.

Lemma 8.3. *Let ψ belong to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and ϕ to the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$ with potential $U \in L^2(\Omega)$ on an open domain $\Omega \subset \mathbb{C}$. If the Willmore functional $\mathcal{W} = 4 \int_{\Omega} U \bar{U} d^2x$*

is minimal with respect to all L^2 -perturbations ΔU with compact support in Ω , which admit a perturbation of ψ and ϕ with compact support in Ω , then there exists spinors $\tilde{\psi}$ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and $\tilde{\phi}$ in the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$, such that (U, \bar{U}) is a complex linear combination of $(\tilde{\phi}_2 \psi_1 + \overline{\tilde{\phi}_1 \psi_2}, \tilde{\phi}_1 \psi_2 + \overline{\tilde{\phi}_2 \psi_1})$ and $(\phi_2 \tilde{\psi}_1 + \overline{\phi_1 \tilde{\psi}_2}, \phi_1 \tilde{\psi}_2 + \overline{\phi_2 \tilde{\psi}_1})$.

Proof. If the support of ΔU is contained in the open subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$, then, due to the quaternionic version of *Cauchy's Integral Formula* 2.1, the restriction of ϕ and ψ to the complement of the domain of $\overline{\Omega'}$ in Ω are uniquely determined by their values on a cycle around Ω and on a cycle in Ω' around the support of U . We conclude that ψ and ϕ admit perturbations with support contained in Ω' , if and only if the total residue with

the corresponding integral kernels vanishes on Ω' . Again due to the quaternionic version of *Cauchy's Integral Formula* 2.1 this is equivalent to the condition that for all elements $\tilde{\psi}$ in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U}-\Delta\bar{U} \\ U+\Delta U & \partial \end{pmatrix}$ and all $\tilde{\phi}$ in the kernel of $\begin{pmatrix} \bar{\partial} & U+\Delta U \\ -\bar{U}-\Delta\bar{U} & \partial \end{pmatrix}$ the residues of the forms $\tilde{\phi}^t \begin{pmatrix} d\bar{z} & 0 \\ 0 & dz \end{pmatrix} \psi$ and $\phi^t \begin{pmatrix} d\bar{z} & 0 \\ 0 & dz \end{pmatrix} \tilde{\psi}$ on Ω' vanish. Due to the equations

$$\begin{aligned} d \left(\tilde{\phi}^t \begin{pmatrix} d\bar{z} & 0 \\ 0 & dz \end{pmatrix} \psi \right) &= \tilde{\phi}^t \begin{pmatrix} 0 & \Delta\bar{U} \\ \Delta U & 0 \end{pmatrix} \psi dz \wedge d\bar{z} \\ d \left(\phi^t \begin{pmatrix} d\bar{z} & 0 \\ 0 & dz \end{pmatrix} \tilde{\psi} \right) &= \phi^t \begin{pmatrix} 0 & \Delta\bar{U} \\ \Delta U & 0 \end{pmatrix} \tilde{\psi} dz \wedge d\bar{z} \end{aligned}$$

this is equivalent to the equations

$$\int_{\Omega} \tilde{\phi}^t \begin{pmatrix} 0 & \sqrt{-1}\Delta\bar{U} \\ \sqrt{-1}\Delta U & 0 \end{pmatrix} \psi d^2x = 0 \quad \int_{\Omega} \phi^t \begin{pmatrix} 0 & \sqrt{-1}\Delta\bar{U} \\ \sqrt{-1}\Delta U & 0 \end{pmatrix} \tilde{\psi} d^2x = 0.$$

We shall apply the implicit function theorem and conclude that the space of perturbations ΔU , which admit perturbations of ψ and ϕ with compact support are submanifolds. Since the question is local, we may chose the domain Ω' to be the unit disk \mathbb{D} . Indeed, appropriate small neighbourhoods of any point are Möbius transforms of \mathbb{D} . On \mathbb{D} we introduce the Banach spaces $L^q(\mathbb{D}, (1 - |z|^2)^{-s} d^2x, \mathbb{H})$ of quaternionic valued L^q -functions with respect to the measure $(1 - |z|^2)^{-s} d^2x$ on the unit disk $\mathbb{D} \subset \mathbb{C}$ with $0 \leq s < 1$.

As a preparation we claim that the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ considered as a closed subspace of $L^p(\mathbb{D}, (1 - |z|^2)^{-s} d^2x, \mathbb{H})$ is contained in $\bigcap_{q < \frac{2p}{2-s}} L^q(\mathbb{D}, \mathbb{H})$. For the proof we apply the quater-

nionic version of *Cauchy's Integral Formula* 2.1. Due to Lemma 2.3 it suffices to show that the integral along the boundary of \mathbb{D} over the integral kernel of $\mathbf{l}_{\mathbb{D}}(0)$ defines a bounded operator from $L^p(\mathbb{D}, (1 - |z|^2)^{-s} d^2x, \mathbb{H})$, into $L^{\frac{2p}{2-s}}(\mathbb{D}, \mathbb{H})$. Due to Young's inequality [R-S-II, Section IX.4 Example 1] the convolution with the function $\frac{1}{\pi z}$ defines an operator from the L^p -functions on the circle $|z'| = r'$ into the L^q -functions on the circle $|z| = r$ with $0 \leq r < r' \leq 1$, which is bounded by

$$\frac{1}{\pi} \left(\int_{\varphi \in \mathbb{R}/2\pi\mathbb{Z}} \frac{1}{\left| 1 - \frac{r}{r'} \exp(2\pi\sqrt{-1}\varphi) \right|^{\frac{pq}{p+pq-q}}} \right)^{1+\frac{1}{q}-\frac{1}{p}},$$

with $1 \leq p < q \leq \infty$. Due to [Ru, 1.4.10.Proposition] this norm is bounded by $C \left| 1 - \frac{r^2}{r'^2} \right|^{\frac{1}{q}-\frac{1}{p}}$. The norm of the restriction of this function to $r \in [0, r_0] \subset [0, 1]$ in the L^q -space on $r \in [0, 1]$ with respect to the measure rdr is bounded by $C' |r' - r_0|^{\frac{2}{q}-\frac{1}{p}} + C''$ with appropriate constants $C' > 0$ and $C'' > 0$. If p' denotes the dual exponent of p with $\frac{1}{p} + \frac{1}{p'} = 1$, then, due to Hölder's

inequality [R-S-I, Theorem III.1 (c)], we obtain for all $f \in L^p \left([0, 1], \frac{r' dr'}{(1-r'^2)^s} \right)$

$$\begin{aligned} \frac{1}{1-r_0} \int_{r_0}^1 f(r') |r' - r_0|^{\frac{2}{q} - \frac{1}{p}} r' dr' &\leq \frac{\|f\|_{L^p \left([0, 1], \frac{r' dr'}{(1-r'^2)^s} \right)}}{1-r_0} \left\| |1-r'^2|^s |r' - r_0|^{\frac{2}{q} - \frac{1}{p}} \right\|_{L^{p'}([r_0, 1], r' dr')} \\ &\leq \|f\|_{L^p \left([0, 1], \frac{r' dr'}{(1-r'^2)^s} \right)} \mathbf{O} (1-r_0)^{\frac{s}{p} + \frac{2}{q} - \frac{2}{p}}. \end{aligned}$$

With $1 \leq q \leq \frac{2p}{2-s}$ this expression remains bounded in the limit $r_0 \rightarrow 1$. Consequently, for $1 \leq q < \frac{2p}{2-s}$ the natural inclusion of the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ in $L^p(\mathbb{D}, (1-|z|^2)^{-s} d^2x, \mathbb{H})$, into $L^q(\mathbb{D}, \mathbb{H})$ is bounded.

Let ψ belong to the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ without zeros on an open neighbourhood $\Omega \subset \mathbb{C}$ of the closed unit disk \mathbb{D} with a small $U \in L^2(\mathbb{D})$. In a second step we claim that for $2 < q < \infty$ and $0 < s < \frac{q-2}{q-1}$ the subset of all ΔU in the Banach space $L^q(\mathbb{D}, (1-|z|^2)^{-s} d^2x)$, such that

$$\int_{\Omega} \tilde{\phi}^t \begin{pmatrix} 0 & \sqrt{-1} \Delta \bar{U} \\ \sqrt{-1} \Delta U & 0 \end{pmatrix} \psi d^2x = 0$$

vanishes for all $\tilde{\phi}$ in the kernel of $\begin{pmatrix} \bar{\partial} & U+\Delta U \\ -\bar{U}-\Delta \bar{U} & \partial \end{pmatrix}$ is a Banach submanifold. In fact, due to the implicit function theorem [R-S-I, Theorem S.11] we have to show that for small ΔU these kernels considered as subspaces of the dual Banach spaces of $\Delta U \in L^q(\mathbb{D}, (1-|z|^2)^{-s} d^2x)$ with respect to the pairing

$$(\tilde{\phi}, \Delta U) \mapsto \int_{\Omega} \tilde{\phi}^t \begin{pmatrix} 0 & \sqrt{-1} \Delta \bar{U} \\ \sqrt{-1} \Delta U & 0 \end{pmatrix} \psi d^2x$$

are isomorphic. Since ψ^{-1} belongs to $\bigcap_{r < \infty} L^r(\mathbb{D}, \mathbb{H}) = \bigcap_{r < \infty} L^r(\mathbb{D}, (1-|z|^2)^{-s} d^2x, \mathbb{H})$ these kernels are contained in $\bigcap_{p < \frac{q}{q-1}} L^p(\mathbb{D}, (1-|z|^2)^{-s} d^2x, \mathbb{H})$. The foregoing claim implies that

these kernels are contained in $\bigcap_{p < \frac{2q}{(q-1)(2-s)}} L^p(\mathbb{D}, \mathbb{H})$. The operator $\mathbf{1} + \mathbf{l}_{\Omega}(U) \begin{pmatrix} 0 & -\Delta \bar{U} \\ \Delta U & 0 \end{pmatrix}$ maps

these kernels onto the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$. If q and s satisfies $\frac{2-s}{2} \left(1 - \frac{1}{q}\right) + \frac{1}{q} - \frac{1}{2} + s \left(1 - \frac{1}{q}\right) < 1 - \frac{1}{q}$, then the operator $\mathbf{l}_{\Omega}(U) \begin{pmatrix} 0 & -\Delta \bar{U} \\ \Delta U & 0 \end{pmatrix}$ is a bounded operator from $\bigcap_{p < \frac{2q}{(q-1)(2-s)}} L^p(\mathbb{D}, \mathbb{H})$ into

$L^{\frac{q}{q-1}}(\mathbb{D}, (1-|z|^2)^{-s} d^2x, \mathbb{H})$. Hence for $0 < s < \frac{q-2}{q-1}$ these kernels are isomorphic.

Obviously, the same statement holds, if ψ is replaced by a spinor ϕ without zeroes in the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$ and $\tilde{\phi}$ by spinors in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U}-\Delta \bar{U} \\ U+\Delta U & \partial \end{pmatrix}$. Moreover, due to the considerations of section 2, the intersection of theses two subspaces of the dual of the Banach space $\Delta U \in L^q(\mathbb{D}, (1-|z|^2)^{-s} d^2x)$ is equal to the linear hull of $\tilde{\phi} = \phi$ and $\tilde{\psi} = \psi$. If ψ is a spinor in the kernel of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \partial \end{pmatrix}$ and ϕ a spinor in the kernel of $\begin{pmatrix} \bar{\partial} & U \\ -\bar{U} & \partial \end{pmatrix}$ without zeroes on Ω , then the subspace of all $\Delta U \in L^q(\mathbb{D}, (1-|z|^2)^{-s} d^2x)$, which admit variations with

compact support of ψ and ϕ , are Banach submanifolds. Furthermore the tangent space of this manifold is the orthogonal complement of these kernels with respect to the corresponding pairings. This implies the statement of the Lemma on the complement of the zeroes of ψ and ϕ . Since a L^2 -functions, which vanishes on this complement, vanishes on the whole of Ω , the Lemma is proven. **q.e.d.**

For all $m \in \mathbb{N}$ and $1 < p < 2$ the operator $\mathbf{l}_\Omega(0)$ is a bounded operator from $W^{m-1,p}(\Omega, \mathbb{H})$ onto $W^{m,p}(\Omega, \mathbb{H})$ (compare [St, Chapter V]). Therefore the kernels of $\begin{pmatrix} \bar{\partial} & -\bar{U} \\ U & \bar{\partial} \end{pmatrix}$ belongs to $\bigcap_{q < \infty} W_{\text{loc}}^{m,p}(\Omega, \mathbb{H})$, if the potential U belongs to $\bigcap_{q < \infty} W_{\text{loc}}^{m-1,p}(\Omega, \mathbb{H})$. Therefore Lemma 8.3 implies that local minimizers of the classes (ii)–(iii) belong to $\bigcap_{n \in \mathbb{N}, q < \infty} W_{\text{loc}}^{m,p}(\Omega)$. With the help of the reality condition for immersion into the pure imaginary quaternions $\simeq \mathbb{R}^3$ these arguments carry over to case (i). **q.e.d.**

We do not claim that the minimizers are realized by immersions. They may have branch points. In general they may be compositions of a finite-sheeted branched covering together with an immersion. Finally, we remark that the existence of minimizers was proven by Simon [Si-1, Si-2] on the class of all smooth immersion from a compact orientable surface of genus one into the Euclidean spaces \mathbb{R}^n ($n \geq 3$). Furthermore, Bauer and Kuwert [B-K] extended these arguments to the classes of all smooth immersions from compact orientable surfaces into the Euclidean spaces \mathbb{R}^n ($n \geq 3$). It might be possible to deduce these results for $n = 3$ and $n = 4$ from our results. In fact, since the stereographic projections of the minimal surfaces in S^3 constructed by Lawson [Law] have Willmore functionals less than 8π (compare [Si-1, Si-2]), it would suffice to prove that at the boundary of the moduli spaces \mathcal{M}_g , which contains stable curves with ordinary double points, the Willmore functional is at least equal to 8π . This would follow from [L-Y], if the corresponding conformal mappings preserve these double points. Moreover, with the help of [K-F] our results might be generalized to conformal mappings into higher-dimensional Euclidean spaces.

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